An introduction to Discrete Dynamical systems

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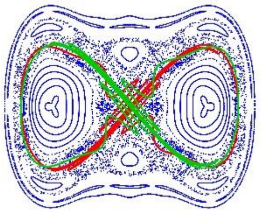

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- Λογισμικό : Mathematica

A simple example

(Peitgen & Richter, 1984)

Deposit Z_0
Rate of interest ε
Period T

$$Z_0, Z_1, Z_2, \ldots.$$

$$Z_{n+1} = (1+\varepsilon)Z_n = (1+\varepsilon)^{n+1} Z_0$$

Prohibition of unlimited wealth $\Rightarrow$ reduction of the rate of interest proportional to Z_n

$$\varepsilon = \varepsilon_0 (1 - Z_n / Z_{\max})$$

$$Z_{n+1} = \varepsilon_0 (1 - Z_n / Z_{\max}) Z_n \qquad \Rightarrow \qquad Z_{n+1} = (1+\varepsilon_0)(1 - x_n) Z_n$$

$$\frac{Z_n}{Z_{\max}} = \frac{1+\varepsilon_0}{\varepsilon} x_n$$

$$r = 1 + \varepsilon_0$$

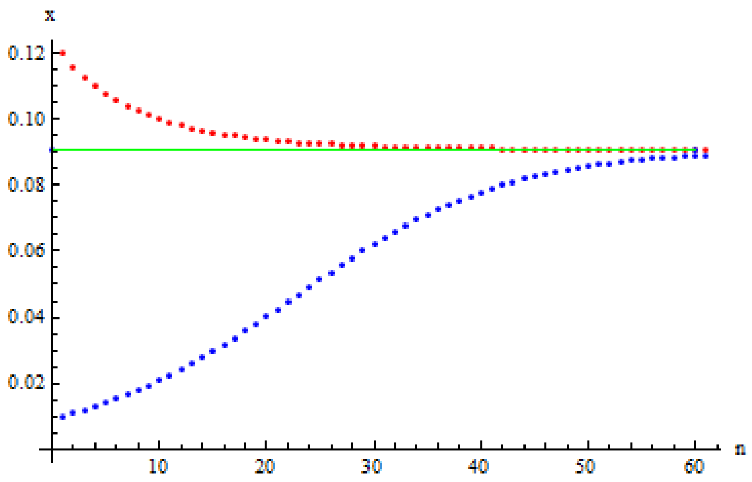
ή

$$\boxed{x_{n+1} = r x_n (1 - x_n)}$$

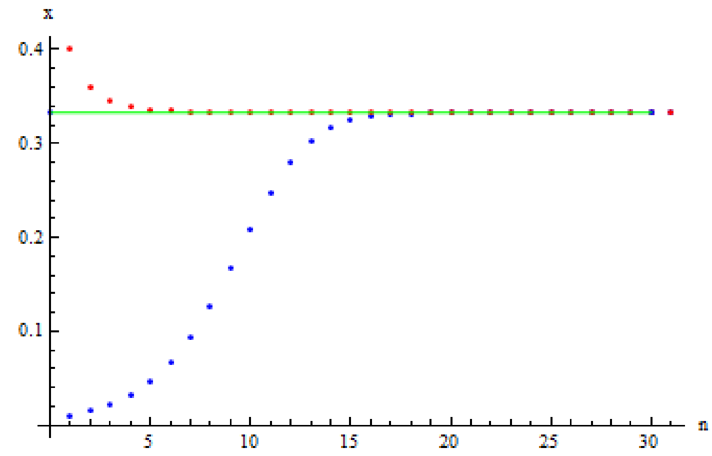
$$x_n = \frac{Z_n}{Z_{\max}} \frac{\varepsilon_0}{1+\varepsilon_0}$$

Logistic map

evolution



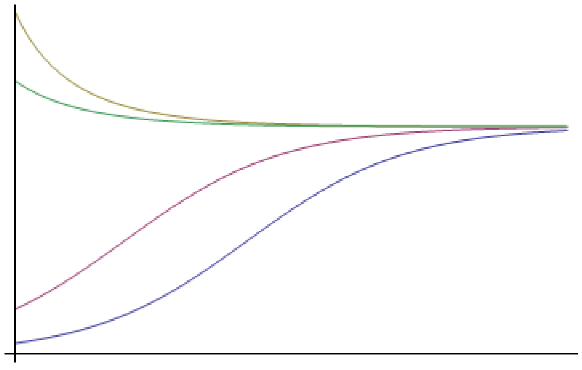
ε_0=0.1,   r=1.1



ε_0=0.5,   r=1.5

continuous system

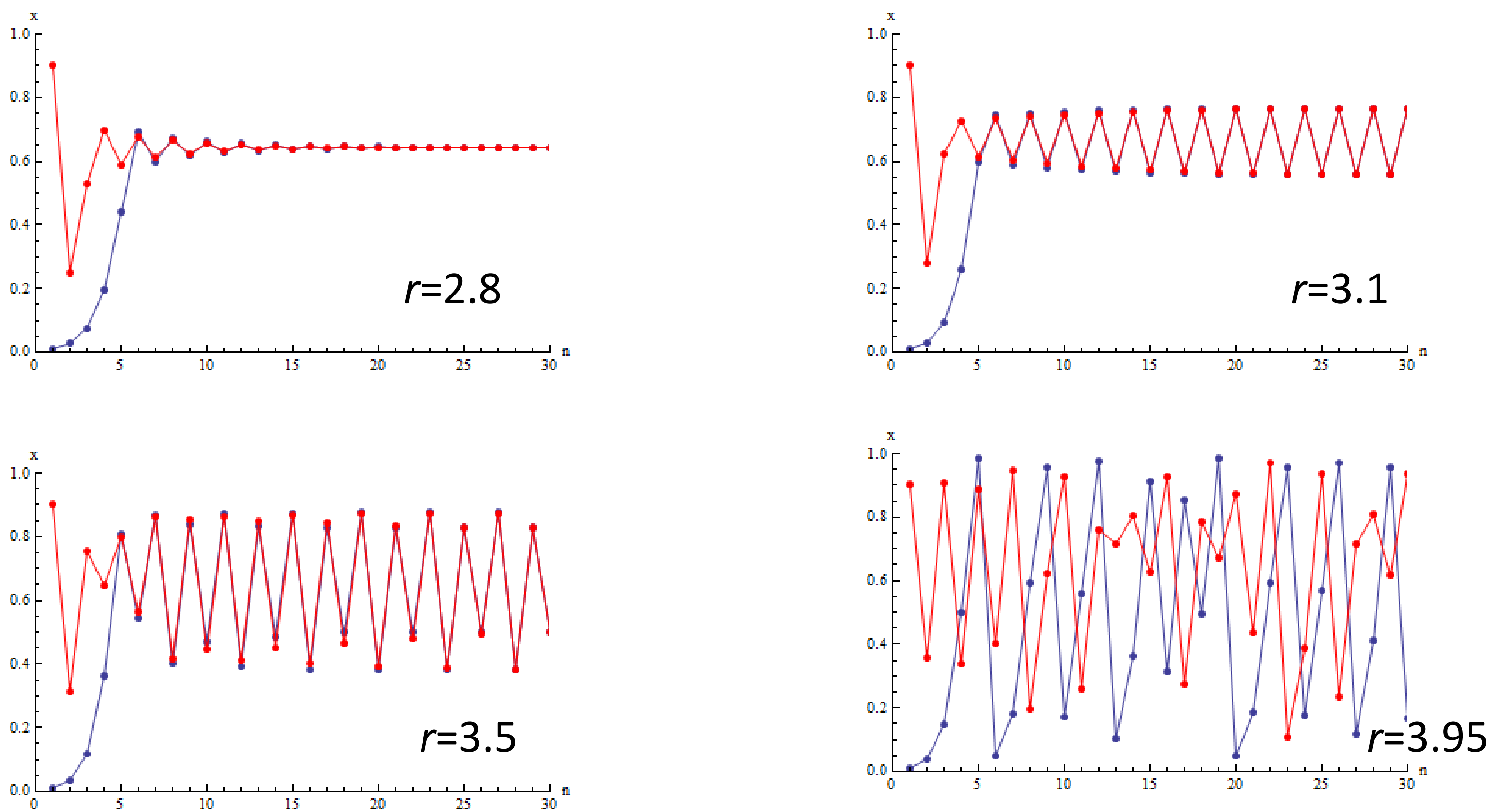
$$\frac{dx}{dt} = r\,x(1-x)$$



evolution – population grow of mayflies

(P.F. Verhulst, 1847, R.M. May, 1976)

$$N_{n+1} = aN_n - rN_n^2 \quad (N_n > 0, \quad \forall n \quad \rightarrow \quad N_{\max} = a/r) \quad \xrightarrow{x=N/N_{\max}} \quad x_{n+1} = rx_n(1-x_n)$$



"Perhaps we would all be better off, not only in research and teaching, but also in every day political and economical life, if more people would take into consideration ***that simple dynamical systems do not necessarily lead to simple dynamical behavior****"*

R.May, "Simple mathematical models with very complicated dynamics", Nature 261, 1976

A. 1D discrete maps

One Dimensional maps

$$x_{n+1} = f(x_n), \qquad x_i \in R, \quad n = 0,1,2,\ldots.$$

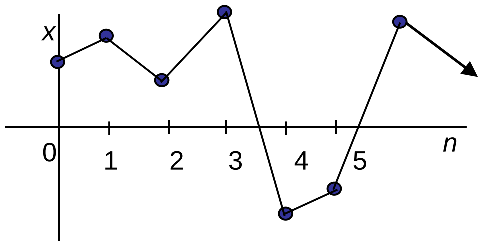
The sequence of values ***T***={x_0 , x_1 , x_2 , ….} is a partial solution or a trajectory of the map that corresponds to the initial value x_0 .

$$x_1 = f(x_0)$$

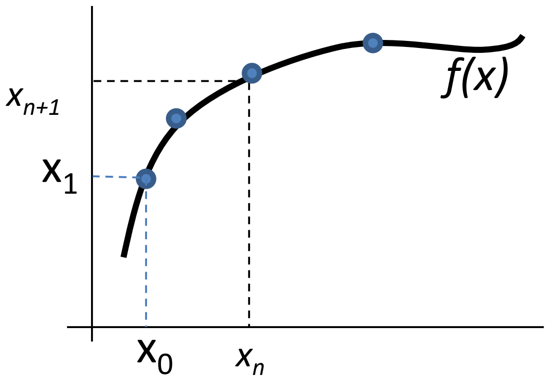
$$x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$$

……………………………………………

$$x_n = f(x_{n-1}) = f(f(\ldots f(x_0)\ldots) = f^n(x_0)$$



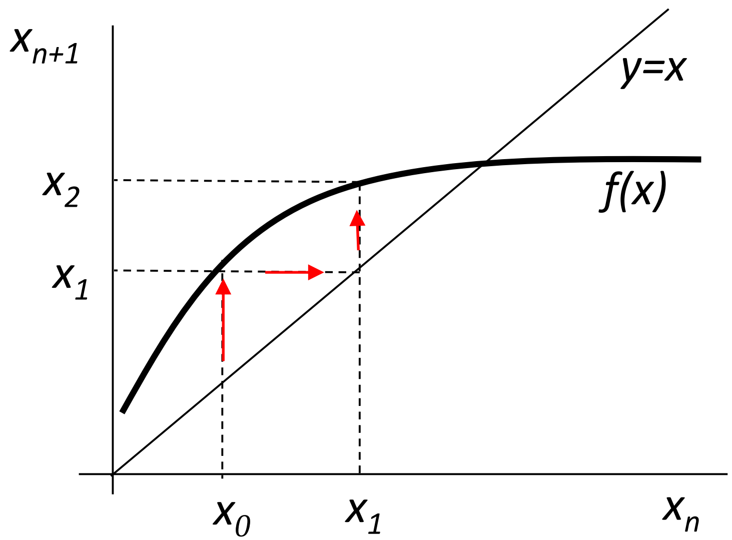
Plane of the map (x_n , x_{n+1})

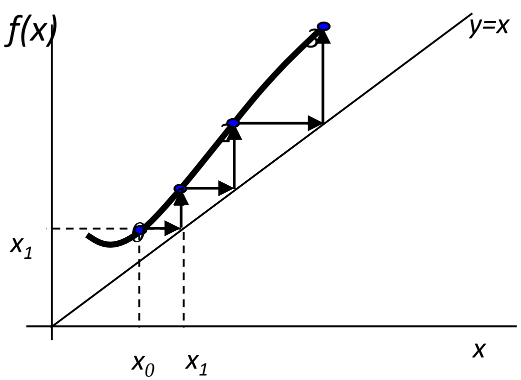


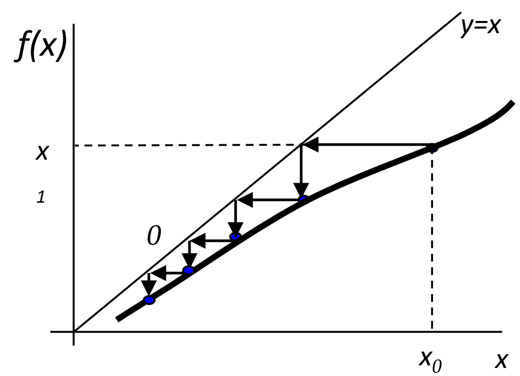
definitions

- The map is called **invertible** if there exist f^{-1} , i.e. $x_n = f^{-1}(x_{n+1})$
  In this case $\boldsymbol{T}$={… x_{-2} , x_{-1} , x_0 , x_1 , x_2 , …}
- An interval $I \subset R$ is **invariant** under the map if $\forall x \in I,\ f(x) \in I$

Graphical presentation (cobweb)







$f(x) > x>0.$

$\lim x_n = \infty \ (n\to\infty)$

$0<f(x) < x.$

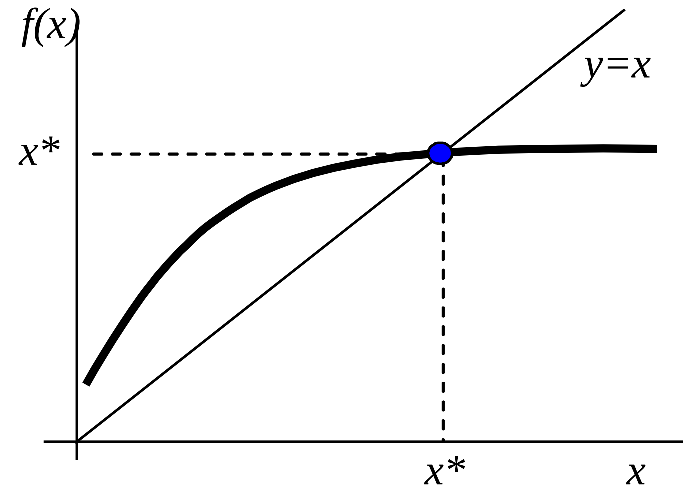
$\lim x_n=0 \ (n\to\infty)$

Special solutions

- Equilibrium solution or **fixed point**

$$x_n = x^*, \qquad \forall n$$

$$\boxed{f(x^*) - x^* = 0}$$



- **Periodic orbits** of period *k*

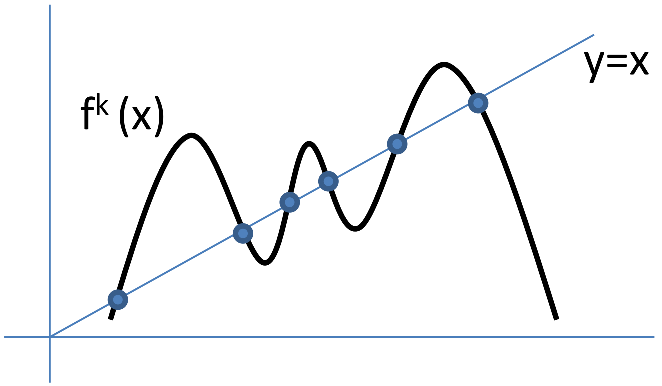
$$x_{n+k} = x_n, \quad \forall n \quad \Rightarrow \quad T^{(k)} = \{x_0, x_1, \ldots, x_{k-1}, x_0, x_1, \ldots\}$$

$$\text{if } \; x^* \in T^{(k)} \quad \Rightarrow \quad x^* = f^k(x^*)$$

$$f^k(x) = \underbrace{f(f(f\ldots f(x)\ldots)))}_{k-times}$$

$$\boxed{f^k(x^*) - x^* = 0}$$

($m \cdot k$ solutions)



Linear maps

- Constant coefficients

$$x_{n+1} = ax_n + b \qquad \Rightarrow \qquad x_n = \begin{cases} x_0 + bn & \text{if } a = 1 \\ \left(x_0 - \dfrac{b}{1-a}\right)a^n + \dfrac{b}{1-a} & \text{if } a \neq 1 \end{cases}$$

- Variable Coefficients

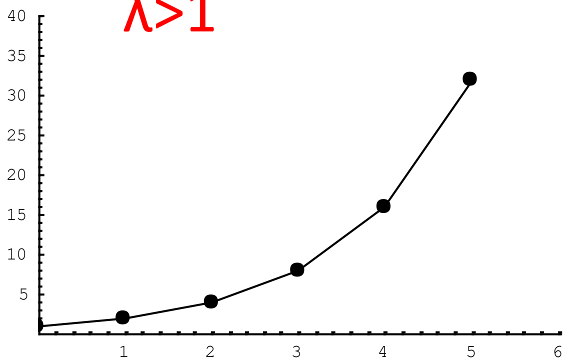
$$x_{n+1} = a_n x_n + b_n \qquad \Rightarrow \qquad x_n = \left(\prod_{i=0}^{n-1} a_i\right) x_0 + \sum_{k=0}^{n-1} \left(\prod_{i=k+1}^{n-1} a_i\right) b_k$$

- Mathematica command : **`RSolve`**

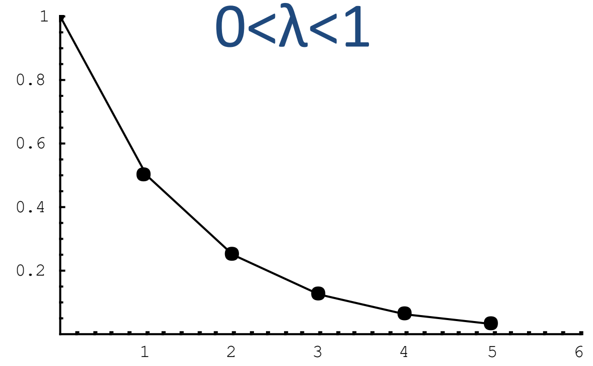
Linear maps

$$x_{n+1} = \lambda x_n \quad (\lambda \in R) \quad \Rightarrow \quad x_n = \lambda^n x_0$$

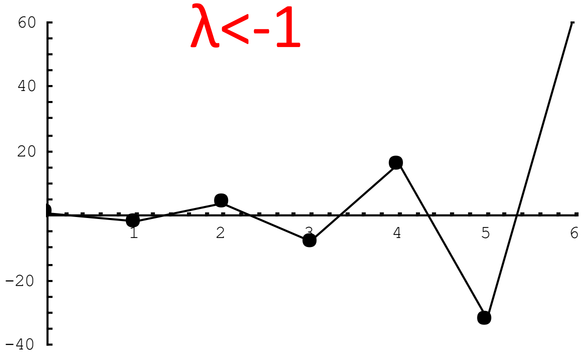
λ>1



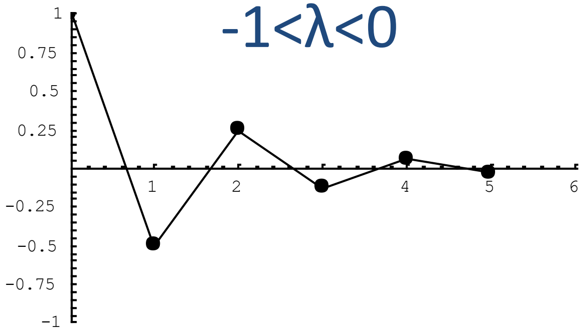
0<λ<1



λ<-1



-1<λ<0



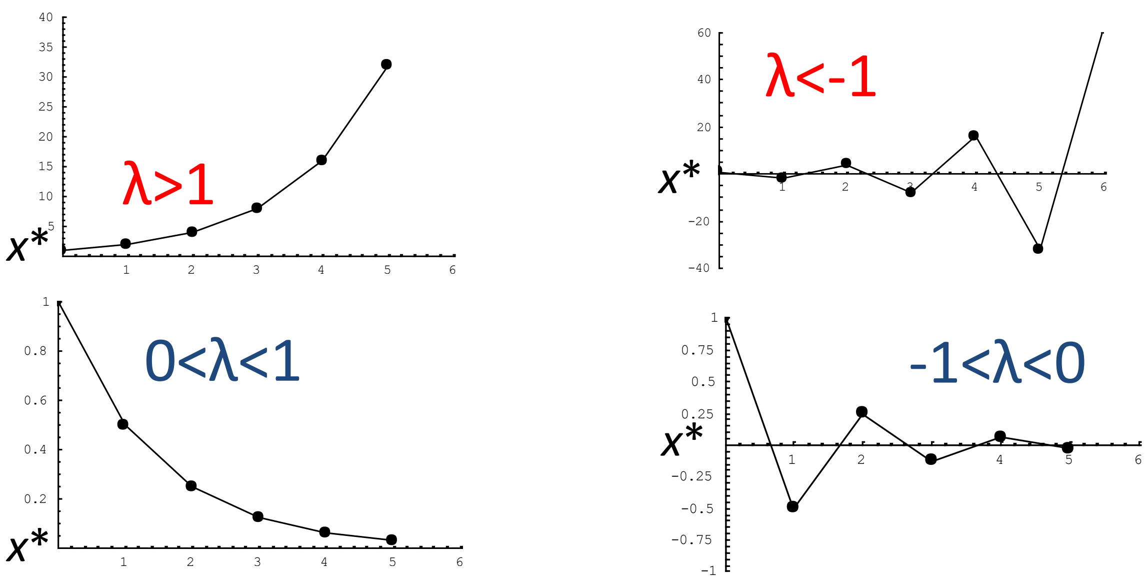
Linearization near Fixed Points

map $x_{n+1} = f(x_n)$

Fixed point $x^* = f(x^*)$

$$x_n = x^* + \delta_n, \qquad x_{n+1} = x^* + \delta_{n+1}, \qquad \delta_n << 1 \qquad \forall n$$

$$x^* + \delta_{n+1} = f(x^* + \delta_n) = f(x^*) + \left.\frac{df}{dx}\right|_{x=x^*} \delta_n + O(\delta_n^2) \quad \Rightarrow \quad \boxed{\delta_{n+1} = \lambda \delta_n, \quad \lambda = \left.\frac{df}{dx}\right|_{x=x^*} = f'(x^*)}$$



Stability

$$\lambda = f\,'(x^*)$$

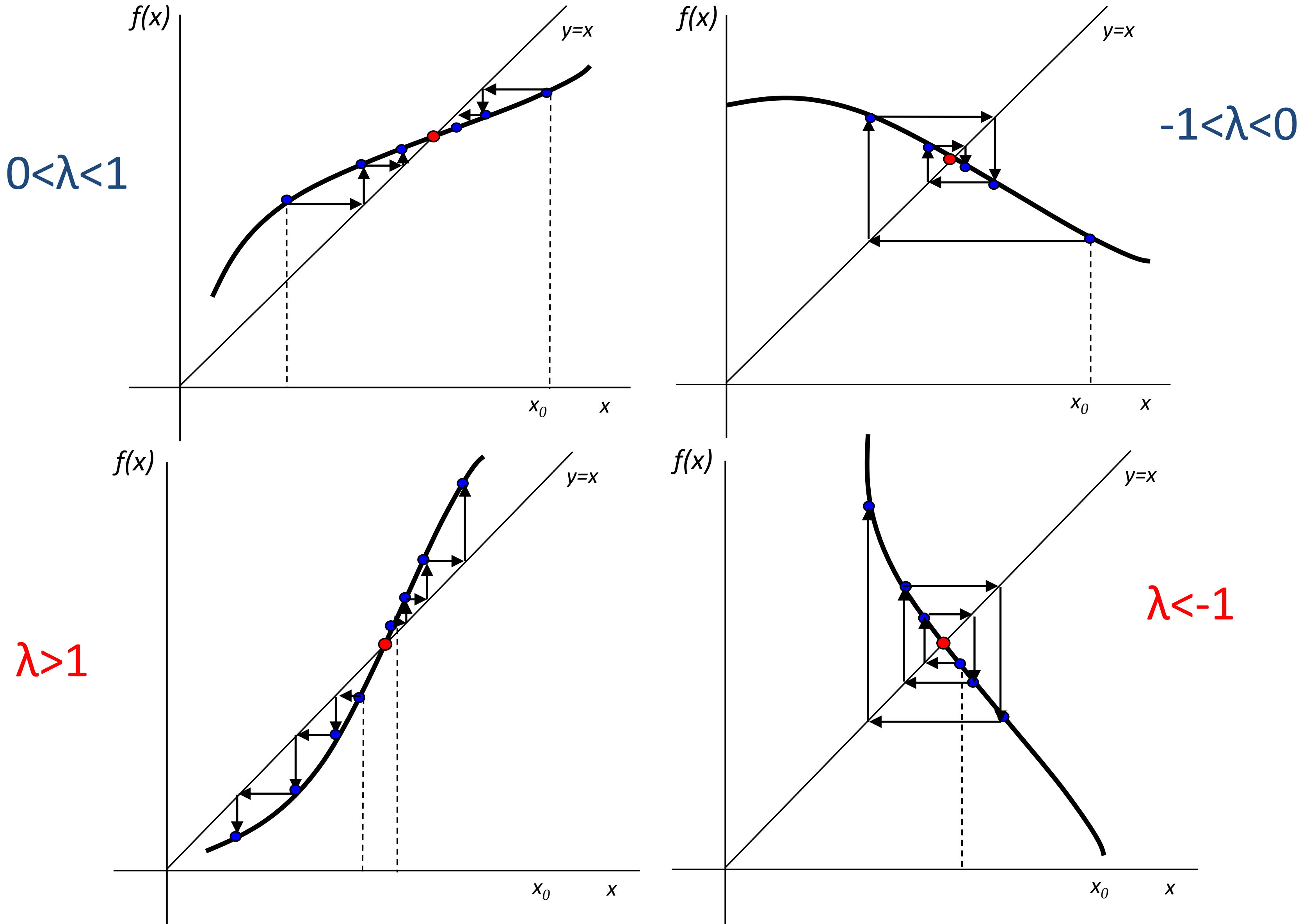
λ>1 : x* is unstable
0<λ<1 : x* is stable

-1<λ<0 : x* is stable with reflection
λ<-1 : x* is unstable

x* = attractor or sink
x* = repeller or source

- If |λ|=1 then x* is **parabolic** or **nonhyperbolic** (critical linear stability)
- If |λ|≠1 then x* is **hyperbolic**

Graphical presentation of stability



Stability in nonhyperbolic case

A. Case $f'(x^*)=1$, $f''(x^*)\neq 0$

- If $f''(x^*)<0$ then x^* is **semi-stable from above**
- If $f''(x^*)>0$ then x^* is **semi-stable from below**

B. Case $f'(x^*)=-1$, $f''(x^*)\neq 0$ [$s_f(x)=2f'''(x)+3(f''(x))^2$]

- If $s_f(x^*)>0$ then x^* is **asymptotically stable**
- If $s_f(x^*)<0$ then x^* is **unstable**

C. Case $f'(x^*)=1$ or -1, $f''(x^*)=0$

$$S_f(x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Schwarzian derivative

- If $S_f(x^*)<0$ then x^* is **asymptotically stable**
- If $S_f(x^*)>0$ then x^* is **unstable**

Kulenovic, M.R.S. & Merino, 2002

Stability of periodic orbits

As for fixed points of the map $\boxed{x_{n+1} = f^{k}(x_n)}$

$$\lambda = \left.\frac{df^{k}}{dx}\right|_{x=x_i^*}, \qquad x_i^* \in T^{(k)} = \{x_0^*, x_1^*, \ldots, x_{k-1}^*\}$$

$$\boxed{\left.\frac{df^{k}}{dx}\right|_{x=x_0} = f'(x_{k-1})\, f'(x_{k-2}) \ldots f'(x_1)\, f'(x_0) = \prod_{i=0}^{k-1} f'(x_i)}$$

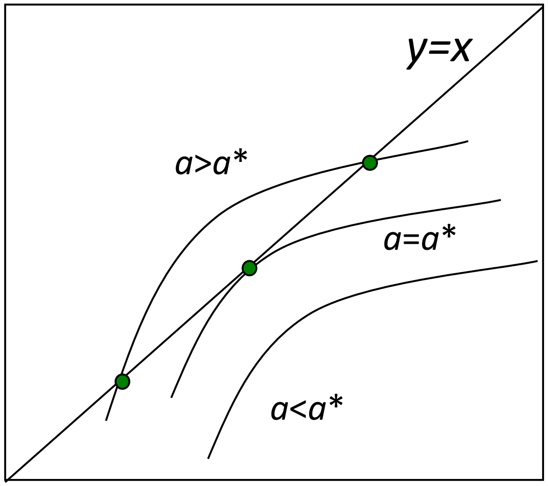
(chain rule of differentiation)

Bifurcations

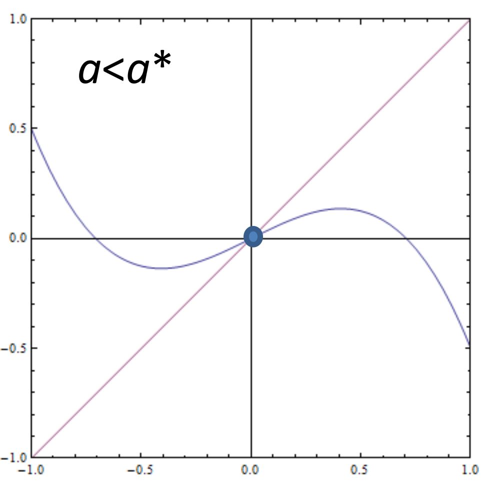
$$x_{n+1} = f(x_n; a), \quad a \in \mathbf{R}$$

parameter

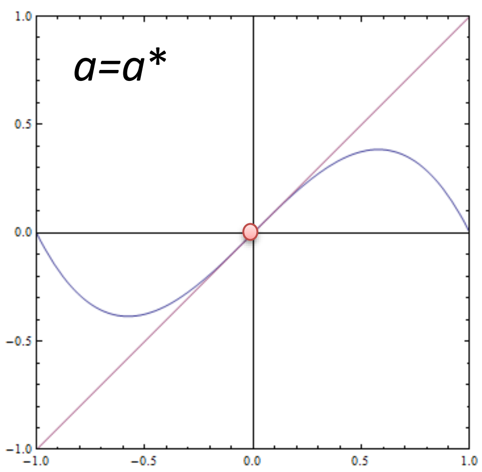
Necessary condition $\color{red}{f'(x^*; a^*) = 1}$



Tangent bifurcation

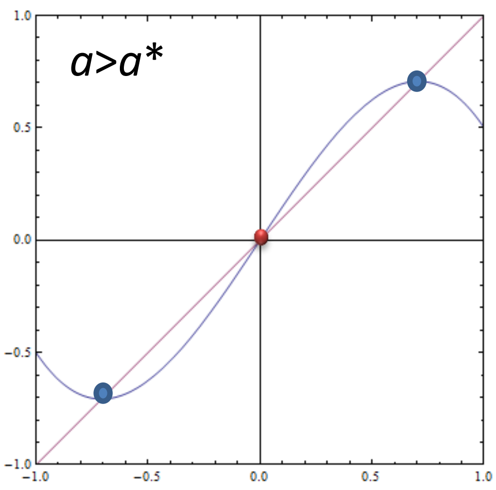


$f'(x^*,a)<1$



$f'(x^*,a)=1$

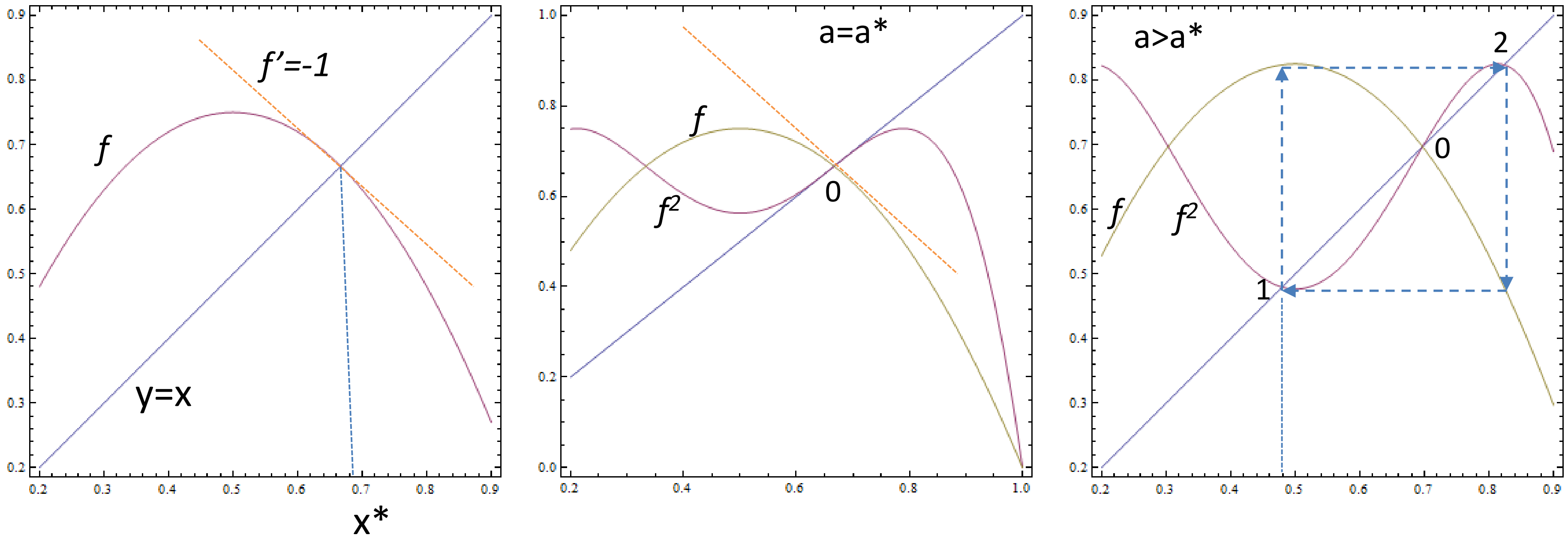
pitchfork bifurcation



$f'(x^*,a)>1$

Period doubling bifurcation

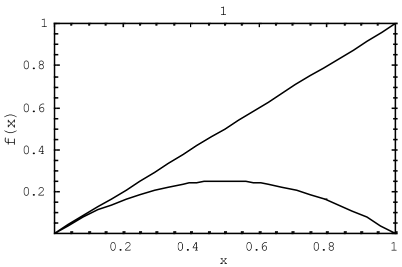
* If $f'(x^*;a^*)=-1$ then $f^{2\prime}(x^*;a^*)=1$ $\Rightarrow$ pitchfork bifurcation with period doubling



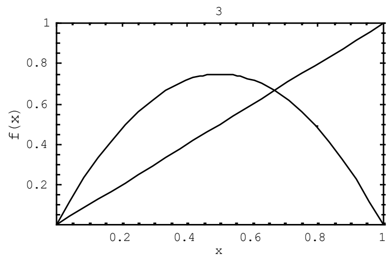
The logistic map

$$f(x) = r\,x(1-x) \qquad , \qquad x_{n+1} = r\,x_n\,(1-x_n), \quad r>0, \quad x_n \in R$$

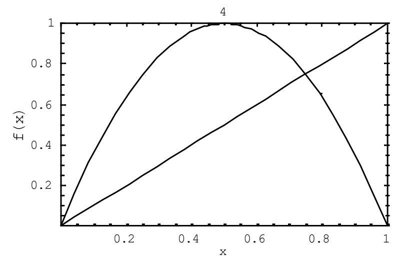
0<*r*<1



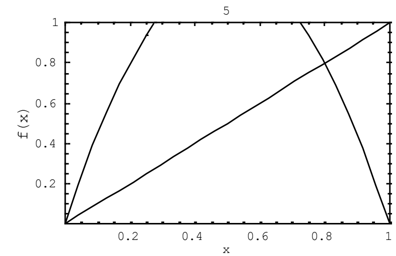
1<*r*<4

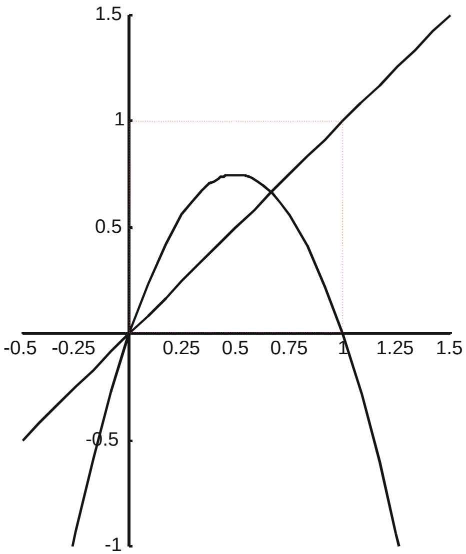


r=4



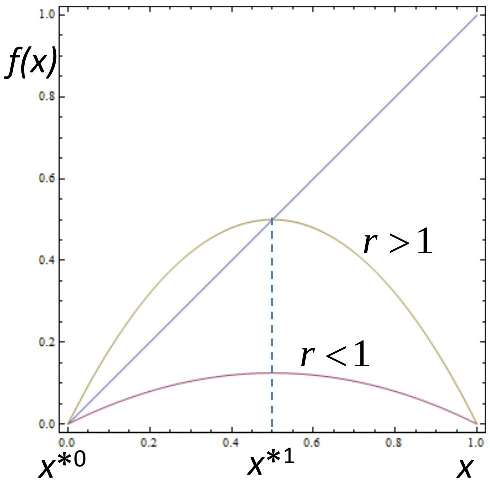
r>4



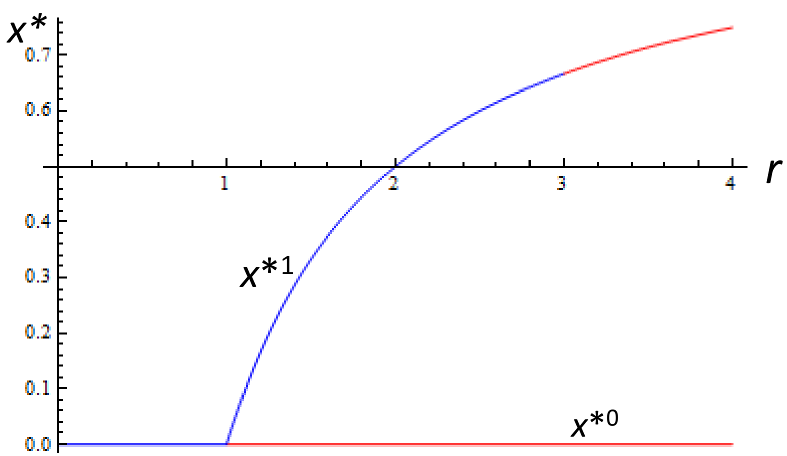


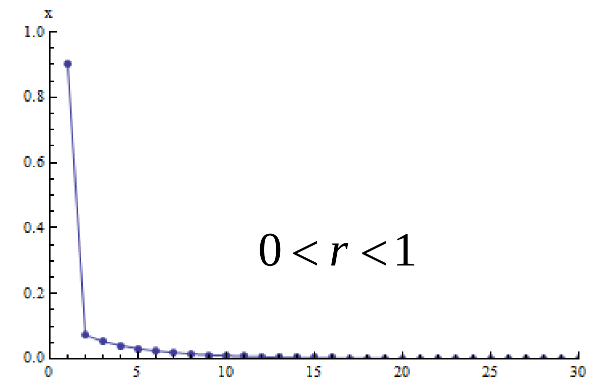
- *f*(*x*) has always a maximum at *x*=1/2 , f(1/2)=r/4
- The interval [0,1] is invariant under *f* for *r*≤4
- For $x_0 < 0$ ή $x_0 > 1 \;\Rightarrow\; \lim_{n\to\infty} x_n = -\infty$
- For *r*>4 there exist open subsets *S*⊂(0,1) such that if $x_0 \in S \;\Rightarrow\; \lim_{n\to\infty} x_n = -\infty$

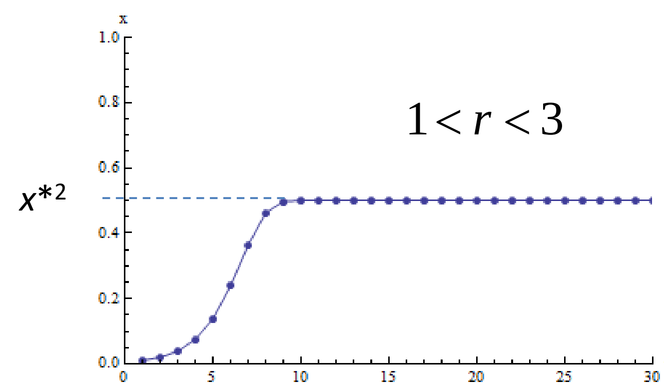
Fixed points



$$x^* = rx^*(1-x^*) \quad \Rightarrow \quad \begin{aligned} x^{*0} &= 0, & f'(x^{*0}) &= r \\ x^{*1} &= \frac{r-1}{r}, & f'(x^{*1}) &= 2-r \end{aligned}$$





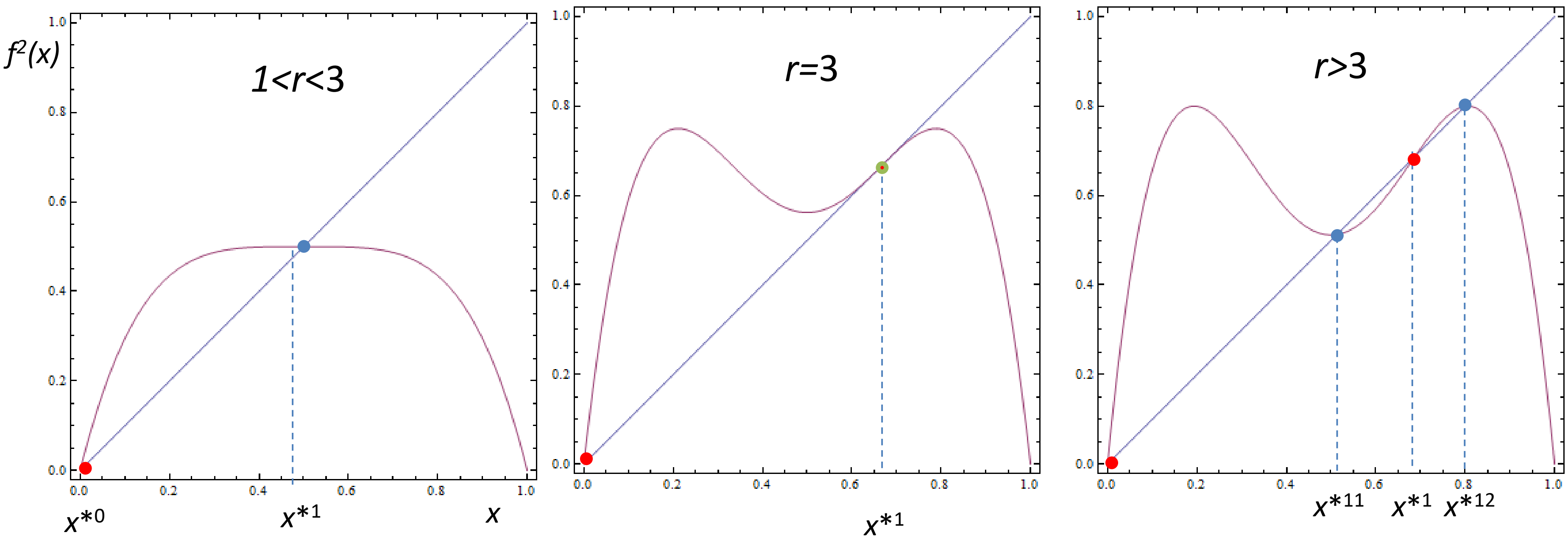


$r > 3$?

Period - 2

$f\,'(x^{*1}) = 2 - r \quad \Rightarrow \quad f\,'(x^{*1})\big|_{r=3} = -1$ ⟶ period doubling ⟶ fixed point for f^2

$$f^{2}(x) = f\,(f\,(x)) = r[rx(1-x)][1-rx(1-x)] = r^{2}x(x-1)(rx^{2}-rx+1)$$



$$f^{2}(x^{*}) = x^{*} \quad \Rightarrow \quad x\,(r-1-rx)\,(r^{2}x^{2}-(r+r^{2})+r+1) = 0$$

x^{*0} x^{*1} x^{*11} x^{*12}

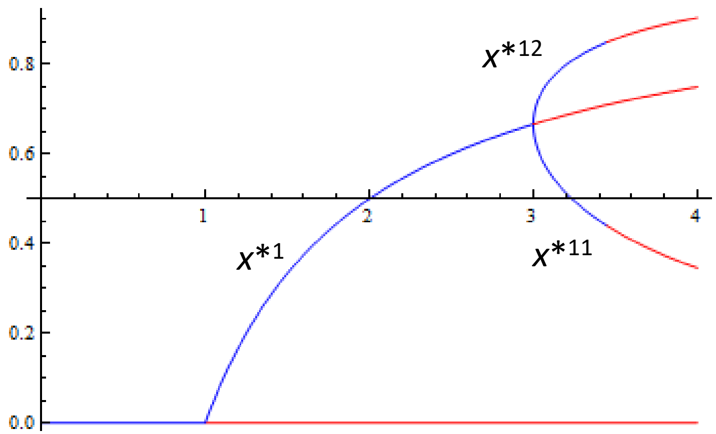
$$\boxed{x^{*11}, x^{*12} = \frac{1+r\pm\sqrt{r^{2}-2r-3}}{2r}}$$

$$x^{*11}, x^{*12} \in R \quad \text{if} \quad r \geq 3$$

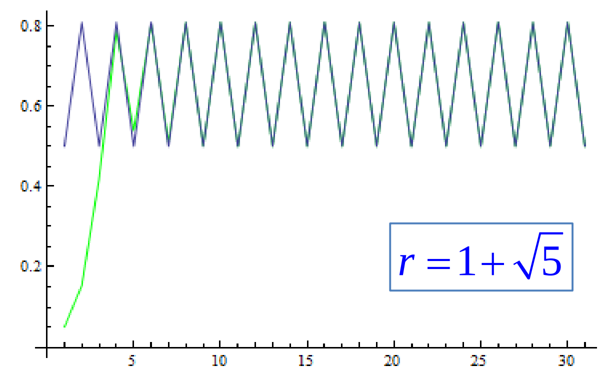
Period – 2 (Stability)

$$\boxed{x^{*11}, x^{*12} = \frac{1+r\pm\sqrt{r^2-2r-3}}{2r}, \quad r>3}$$

$$[f^2(x)']_{x=x^{*11}} = [f^2(x)']_{x=x^{*12}} = 4+2r-r^2 = \begin{cases} |\cdot|<1 & \text{if} \quad 3<r<1+\sqrt{6} \\ |\cdot|>1 & \text{if} \quad r>1+\sqrt{6} \end{cases}$$



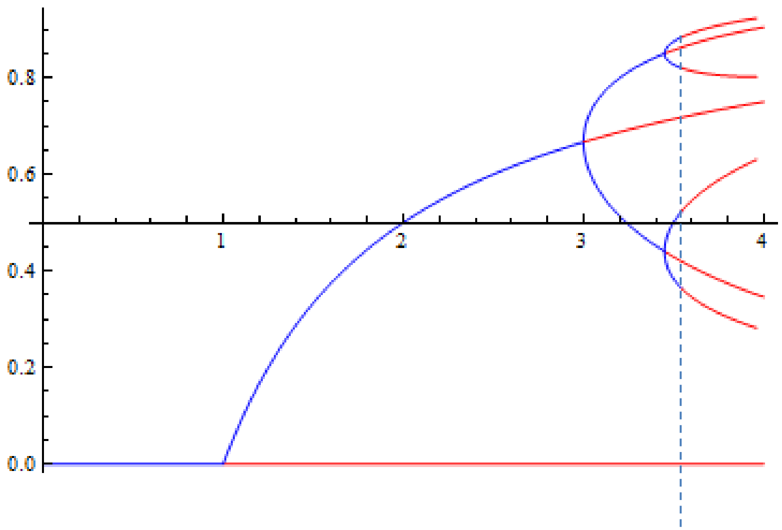
$$r? \quad x^{*11} = \frac{1}{2} \qquad \boxed{\frac{1+r-\sqrt{r^2-2r-3}}{2r} = \frac{1}{2} \quad \Rightarrow \quad r = 1+\sqrt{5}}$$



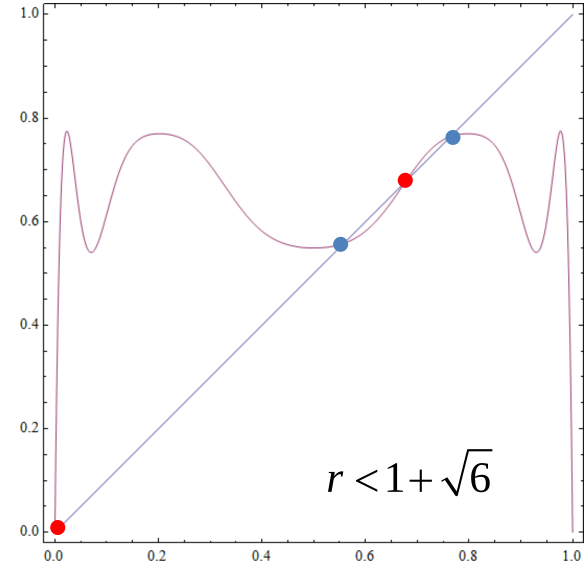
$$\boxed{[f^2(x)']_{x=x^{*12},\, r=r_c} = 4+2r-r^2\Big|_{r=1+\sqrt{6}} = -1}$$ → period doubling → fixed point for f^4

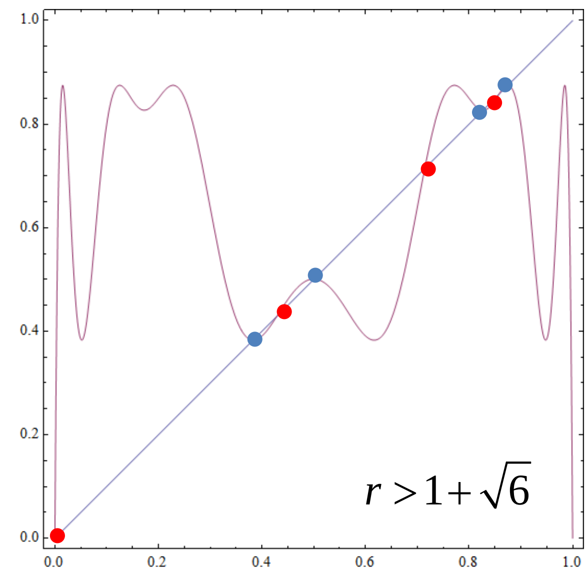
Period – 4

$$f^4(x)= r^4 x - r^4 x^2 - r^5 x^2 - r^6 x^2 - r^7 x^2 + 2 r^5 x^3 + 2 r^6 x^3 + 4 r^7 x^3 + 2 r^8 x^3 + 2 r^9 x^3 -$$
$$r^5 x^4 - r^6 x^4 - 7 r^7 x^4 - 7 r^8 x^4 - 7 r^9 x^4 - 6 r^{10} x^4 - r^{11} x^4 + 6 r^7 x^5 + 10 r^8 x^5 +$$
$$10 r^9 x^5 + 24 r^{10} x^5 + 10 r^{11} x^5 + 4 r^{12} x^5 - 2 r^7 x^6 - 8 r^8 x^6 - 8 r^9 x^6 - 36 r^{10} x^6 -$$
$$36 r^{11} x^6 - 22 r^{12} x^6 - 6 r^{13} x^6 + 4 r^8 x^7 + 4 r^9 x^7 + 24 r^{10} x^7 + 64 r^{11} x^7 +$$
$$52 r^{12} x^7 + 36 r^{13} x^7 + 4 r^{14} x^7 - r^8 x^8 - r^9 x^8 - 6 r^{10} x^8 - 61 r^{11} x^8 - 70 r^{12} x^8 -$$
$$90 r^{13} x^8 - 28 r^{14} x^8 - r^{15} x^8 + 30 r^{11} x^9 + 60 r^{12} x^9 + 120 r^{13} x^9 + 84 r^{14} x^9 +$$
$$8 r^{15} x^9 - 6 r^{11} x^{10} - 34 r^{12} x^{10} - 90 r^{13} x^{10} - 140 r^{14} x^{10} - 28 r^{15} x^{10} + 12 r^{12} x^{11} +$$
$$36 r^{13} x^{11} + 140 r^{14} x^{11} + 56 r^{15} x^{11} - 2 r^{12} x^{12} - 6 r^{13} x^{12} - 84 r^{14} x^{12} -$$
$$70 r^{15} x^{12} + 28 r^{14} x^{13} + 56 r^{15} x^{13} - 4 r^{14} x^{14} - 28 r^{15} x^{14} + 8 r^{15} x^{15} - r^{15} x^{16}$$



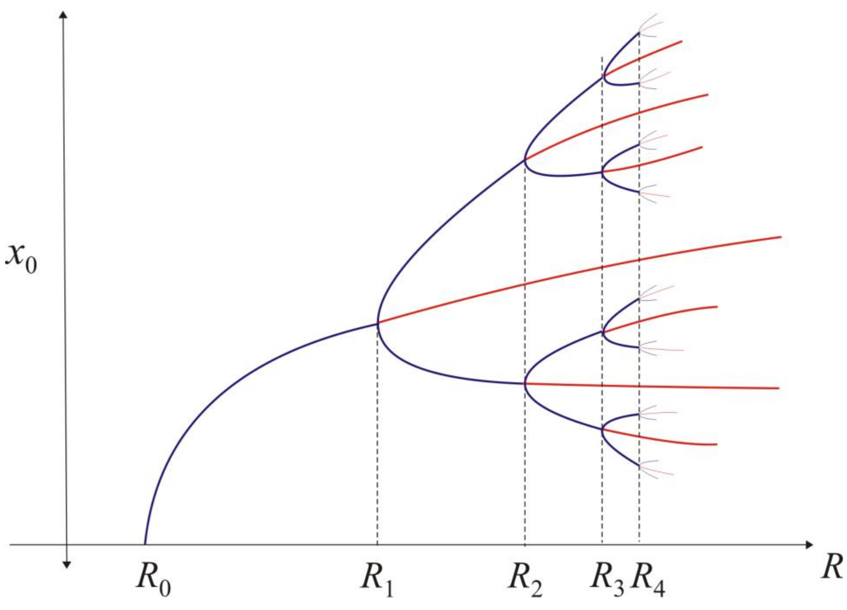
r_c=3.54409

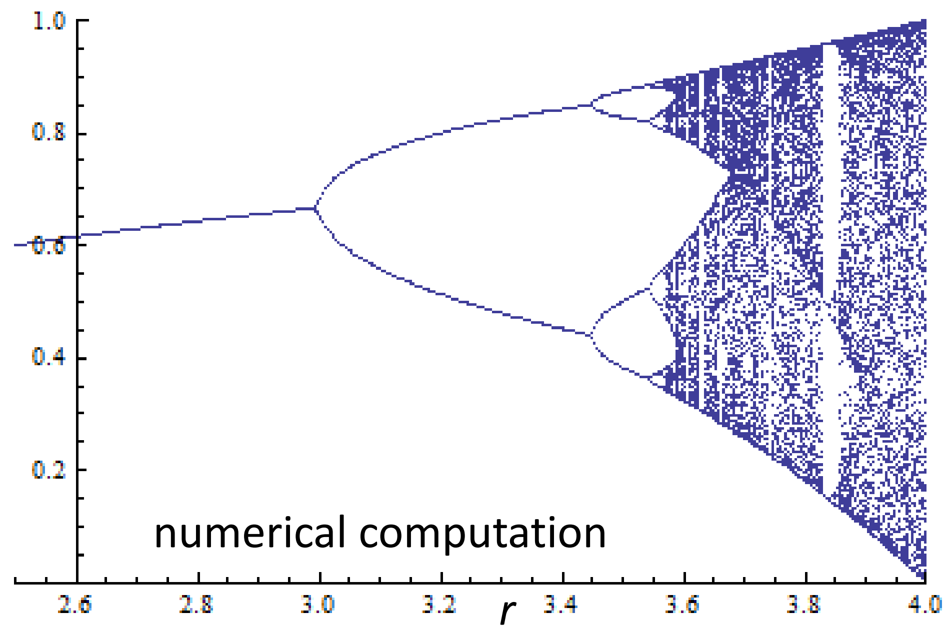




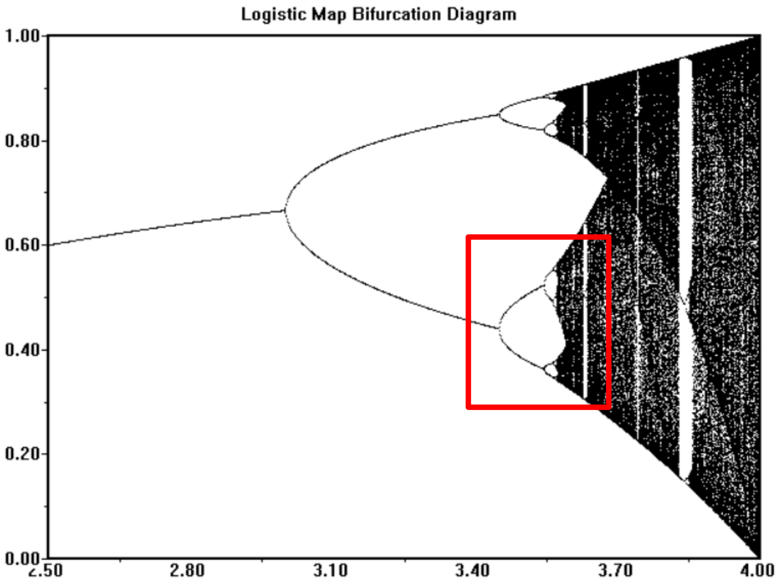
Bifurcation diagram

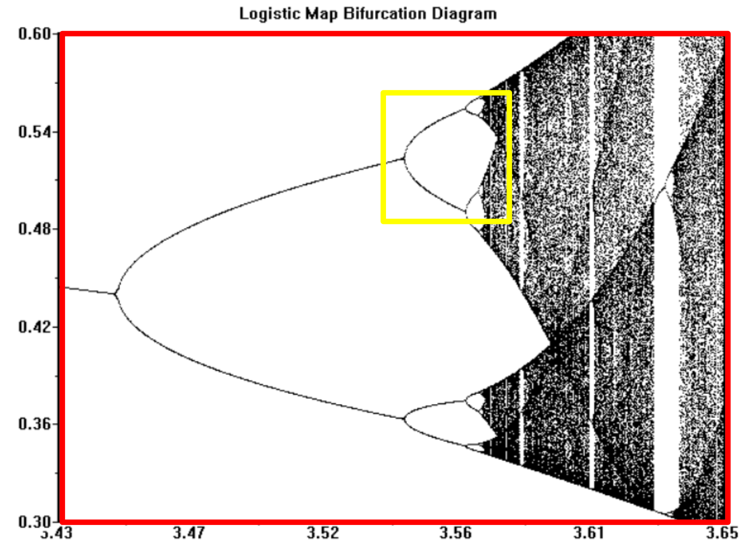
		stable	unstable	R
Fixed point	$x^{*}=0$	$0<r<1$	r>1	
Fixed point	$x^{*}=\dfrac{r-1}{r}$	$1<r<R_1$	$r>R_1$	$R_1=3$
Period-2	$x_{1,2}^{*}=\dfrac{1+r\pm\sqrt{(r-3)(r+1)}}{2r}$	$R_1<r<R_2$	$r>R_2$	$R_2=1+\sqrt{6}\approx 3.4495$
Period-4	numerical	$R_2<r<R_3$	$r>R_3$	$R_3\approx 3.54409$
Period-8	……………….	……………	………..	……………

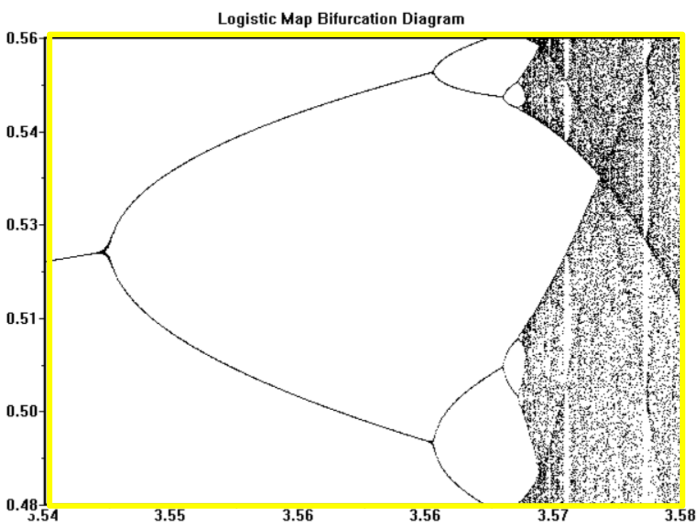




Bifurcation diagram

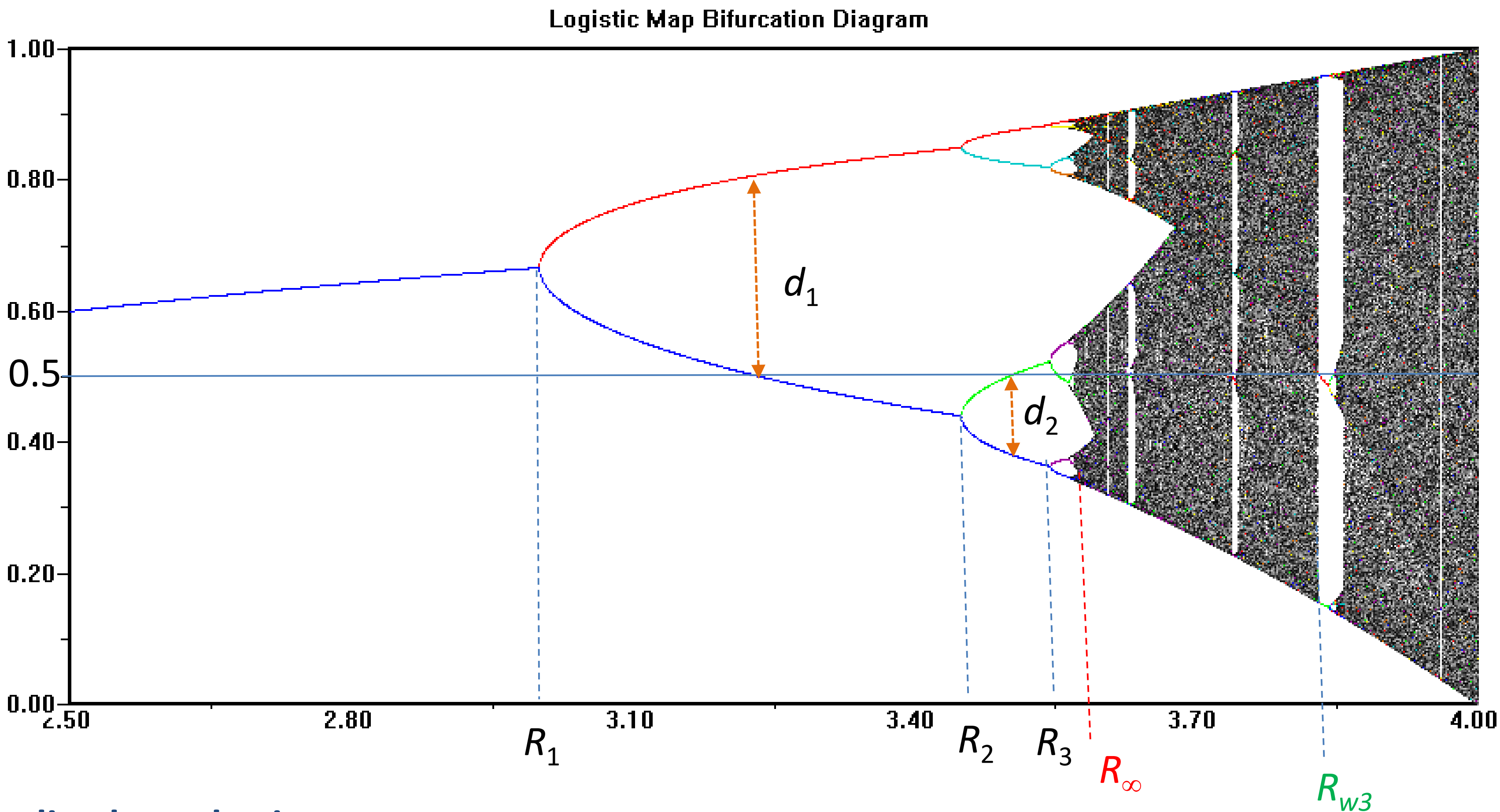






Scaling Laws
(vertically & horizontaly)

(Feigenbaum, 1978)



Scaling hypothesis

$$a \approx \frac{d_n}{d_{n+1}} \approx 2.50290 \quad (n \to \infty)$$

$$R_\infty - R_n \approx const \cdot \delta^{-n} \quad (n \to \infty)$$

$$\delta \approx 4.6692016$$

a, δ : Feigenbaum constants

$$R_\infty \approx 3.5699456$$

Computation Formulas ($n \to \infty$)

$r = \overline{r}_n$: n-period supercycle (x_0 =1/2)

$$d_1 = f\left(\tfrac{1}{2};\overline{r}_1\right) - \frac{1}{2}, \quad d_2 = f^2\left(\tfrac{1}{2};\overline{r}_2\right) - \frac{1}{2}, \quad \ldots \quad , d_n = f^{2^{n-1}}\left(\tfrac{1}{2};\overline{r}_n\right) - \frac{1}{2}$$

$$a \approx -\frac{d_n}{d_{n+1}} \quad \Rightarrow \quad a \approx -\frac{f^{2^{n-1}}\left(\tfrac{1}{2};\overline{r}_n\right) - 1/2}{f^{2^n}\left(\tfrac{1}{2};\overline{r}_{n+1}\right) - 1/2}$$

$$R_\infty - R_n \approx c \cdot \delta^{-n} \qquad (1)$$

$$R_n - R_{n+1} = (R_n - R_\infty) - (R_{n+1} - R_\infty) \overset{(1)}{\approx} -c \cdot \delta^{-n} + c \cdot \delta^{-(n+1)} = -c\,\delta^{-n}(1 - \delta^{-1}) \qquad (2a)$$

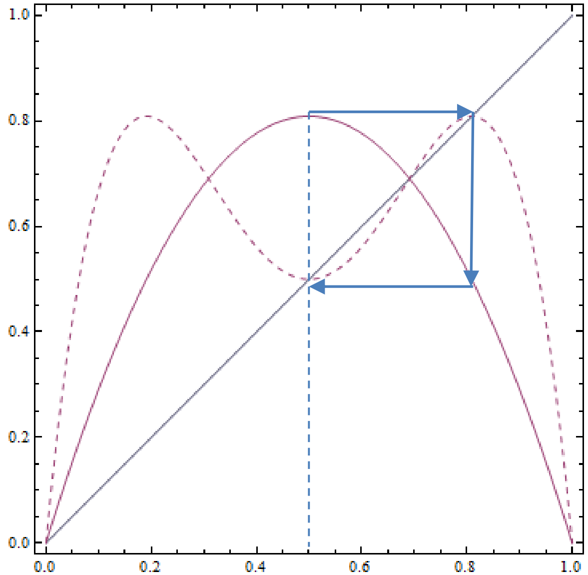
$$R_{n+1} - R_{n+2} = -c\,\delta^{-(n+1)}(1 - \delta^{-1}) \qquad (2b)$$

(2a) & (2b) $\Rightarrow$ $$\delta \approx \frac{R_n - R_{n+1}}{R_{n+1} - R_{n+2}}$$

$$\begin{aligned} R_n - R_\infty &\approx c \cdot \delta^{-n} \\ R_{n+1} - R_\infty &\approx c \cdot \delta^{-(n+1)} \end{aligned} \quad \Rightarrow \quad \frac{R_n - R_\infty}{R_{n+1} - R_\infty} \approx \delta \approx \frac{R_n - R_{n+1}}{R_{n+1} - R_{n+2}} \quad \Rightarrow \quad R_\infty \approx \frac{R_n R_{n+2} - R_{n+1}^2}{R_n - 2R_{n+1} + R_{n+2}}$$

Supercycles

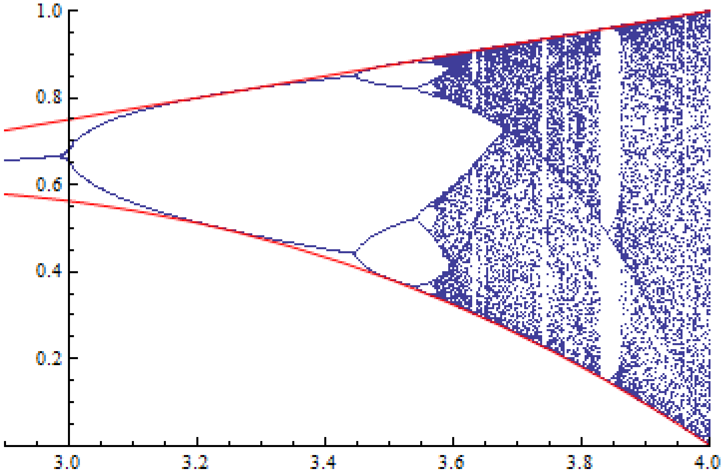
$r = 1 + \sqrt{5}$



x_c=1/2

$$f(1/2) = \frac{r}{4} = \max$$

$$f^2(1/2) = \frac{1}{16}(r-4)r^2 = \min$$



Which values of r provide k-periodic orbits that include x_c=1/2 ?

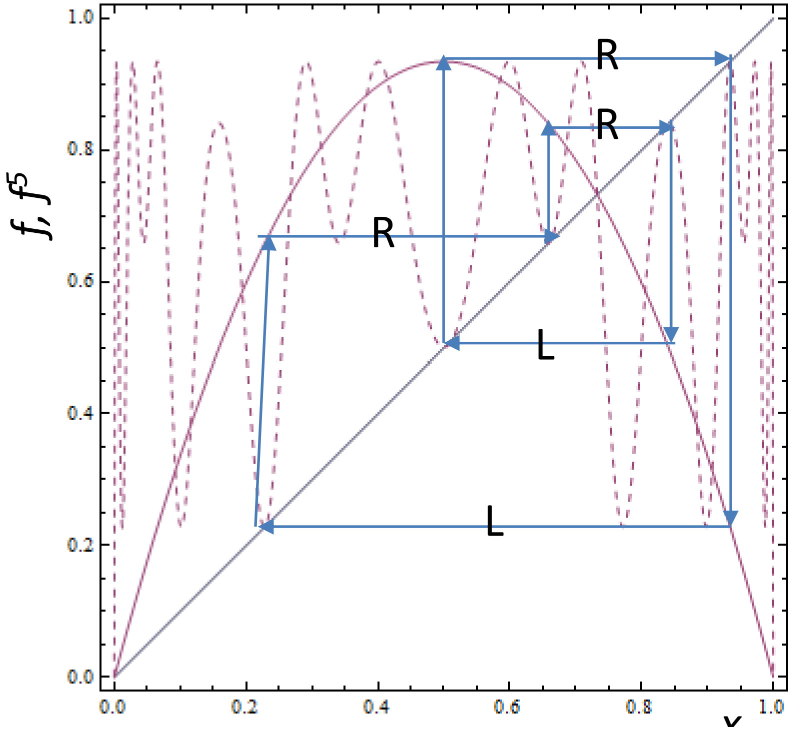
Solve the equation $f^k(x = 1/2, r) = 1/2$ **Supercycle solutions**

Supercycles are the "most stable" periodic orbits

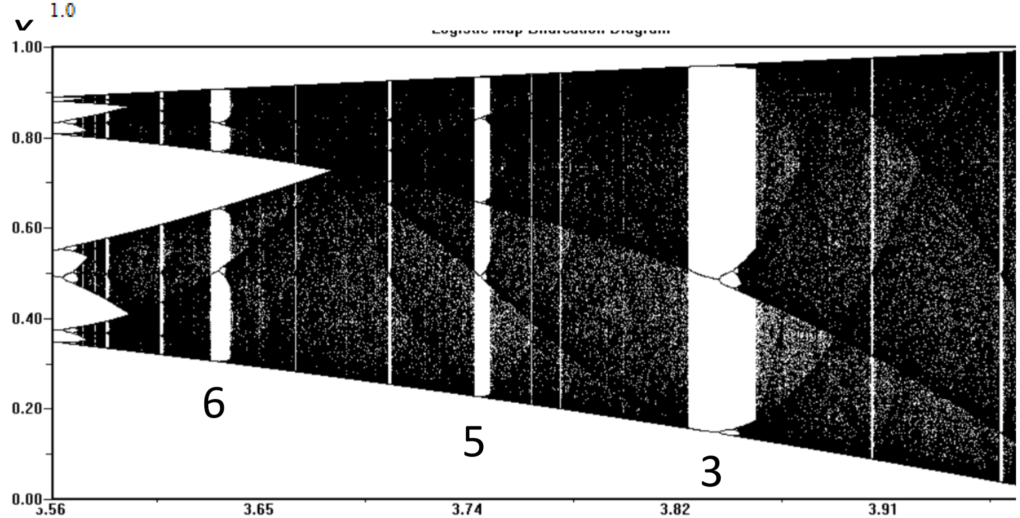
$$f^k(x)'\big|_{x=1/2} = f(x)'\big|_{x_0=1/2} \cdot f(x)'\big|_{x_1} \ldots = 0 \in (-1,1)$$

Supercycles

Metropolis, Stein & Stein (1973)



3.2361	2	RL
3.4986	4	RLRL
3.6275	**6**	**RLRRRL**
3.7389	**5**	**RLRRL**
3.8319	**3**	**RLR**
3.8446	**6**	**RLLRL**
3.9057	**5**	**RLLRL**
....	.....	....



Sharkovsky's ordering

$$3 \rhd 5 \rhd 7 \ldots \quad \rhd 2\times 3 \rhd 2\times 5 \ldots \rhd 2^2\times 3 \rhd 2^2\times 5 \rhd \ldots \quad \rhd 2^n\times 3 \rhd 2^n\times 5 \rhd \ldots \rhd 2^n \rhd \ldots \rhd 2^2 \rhd 2 \rhd 1$$

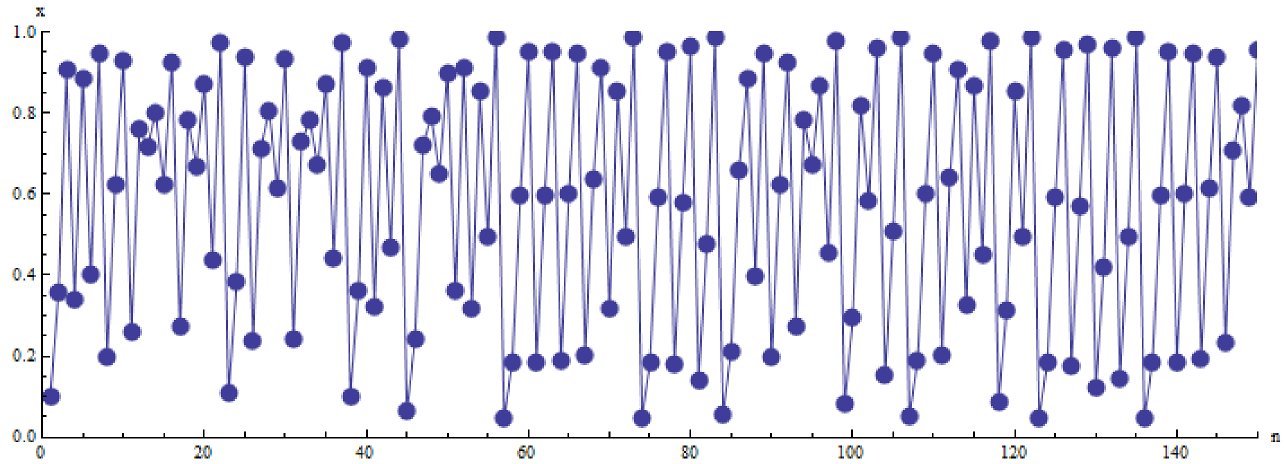
Sharkovsky's theorem

if f has a periodic orbit of period *k*, then it must posses a periodic orbit of period *m*, for all *m* with $k \rhd m$

- The logistic map possesses periodic orbits of all periods of the Sharkovsky's list

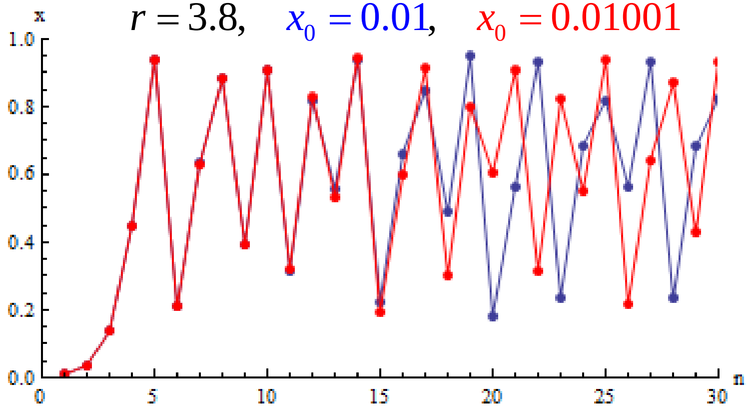
Chaos

$$r \in \left(R_\infty, 4\right) - \bigcup \Delta_i, \qquad \Delta_i = (R_i^{(w)}, R_{i+1}^{(w)})$$

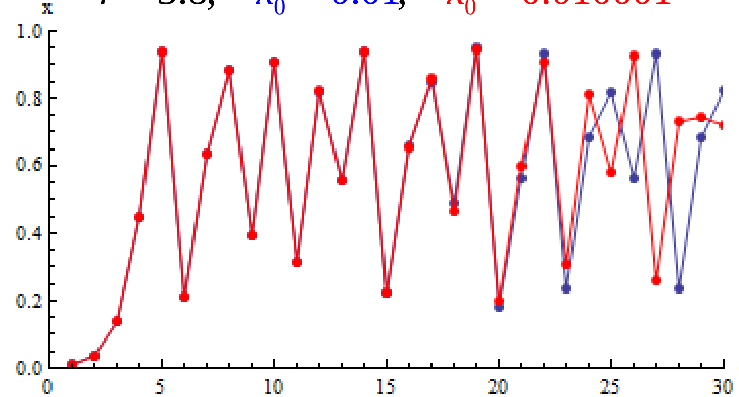


Sensitivity to initial conditions

$r = 3.8, \quad \color{blue}{x_0 = 0.01}, \quad \color{red}{x_0 = 0.01001}$



$r = 3.8, \quad \color{blue}{x_0 = 0.01}, \quad \color{red}{x_0 = 0.010001}$

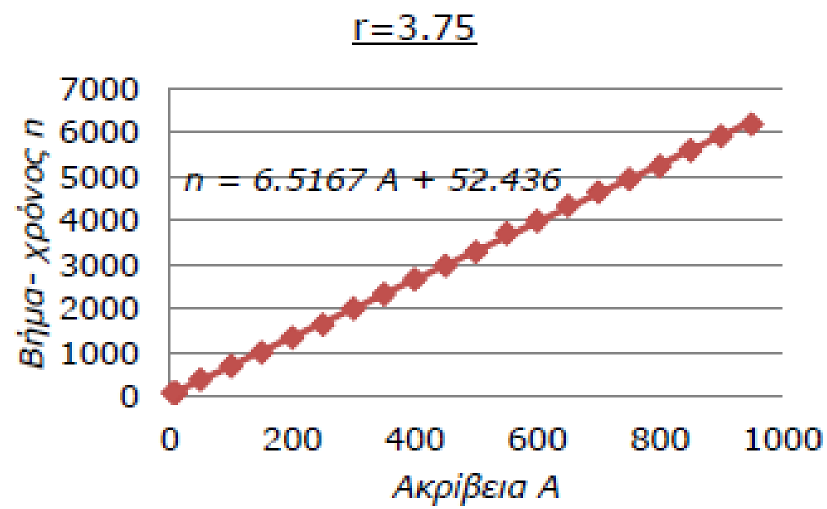


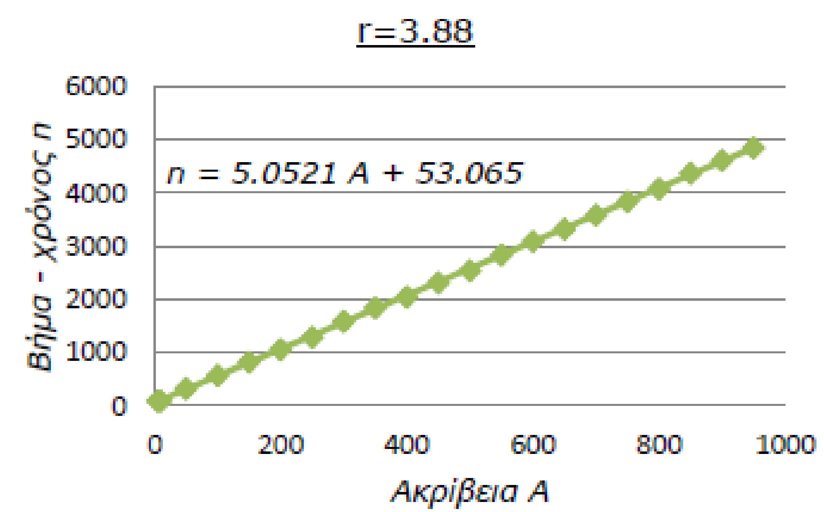
Arbitrary precision computations - 1000 digits

$$y_0 = x_0 + 10^{-A}, \qquad |y_n - x_n| > 0.01$$

Πίνακας Ι: Βήμα-χρόνος που η τροχιά με τη δεδομένη ακρίβεια αποκλίνει α
τροχιά αναφοράς για διάφορες τιμές του r.

r= 3.64		r= 3.75		r= 3.88	
Ακρίβεια	Βήμα - Χρόνος	Ακρίβεια	Βήμα - Χρόνος	Ακρίβεια	Βήμα - Χρόνος
950	9726	950	6186	950	4852
900	9224	900	5913	900	4616
850	8732	850	5590	850	4376
800	8158	800	5233	800	4081
750	7658	750	4944	750	3844
700	7158	700	4645	700	3585
650	6652	650	4334	650	3324
600	6168	600	3992	600	3094
550	5664	550	3710	550	2837
500	5092	500	3294	500	2542
450	4616	450	2978	450	2314
400	4146	400	2665	400	2048
350	3636	350	2329	350	1841
300	3184	300	2005	300	1588
250	2622	250	1643	250	1289
200	2146	200	1345	200	1058
150	1664	150	1006	150	824
100	1160	100	702	100	570
50	698	50	400	50	322
10	194	10	97	10	90
5	188	5	97	5	90





Lyapunov exponent

$$x_N = f^N(x_0) \quad , \quad x'_N = f^N(x_0+\varepsilon) \quad , \quad \lambda = \lambda(x_0), \quad \varepsilon << 1$$

$$\varepsilon e^{\lambda N} = | f^N(x_0+\varepsilon) - f^N(x_0) | \quad \Rightarrow \quad \lambda = \frac{1}{N}\log\left|\frac{f^N(x_0+\varepsilon) - f^N(x_0)}{\varepsilon}\right|$$

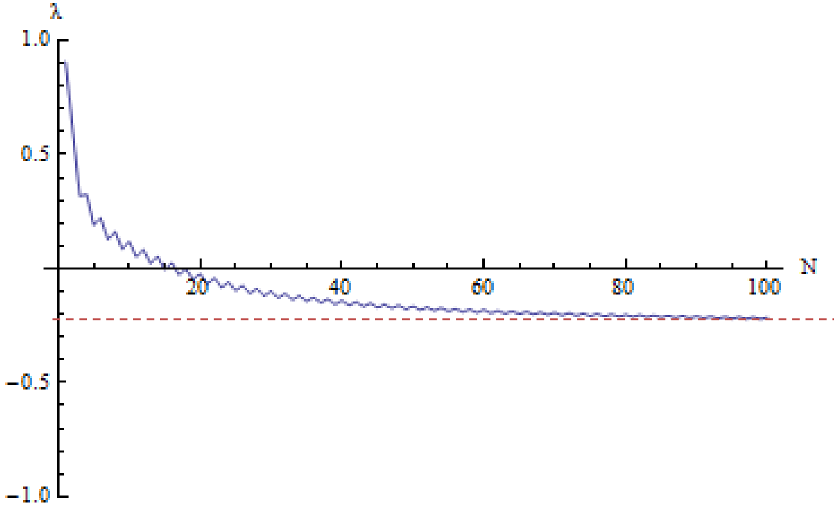
For $\varepsilon \to 0$ and $N \to \infty$

$$\lambda = \lim_{N\to\infty}\frac{1}{N}\log\left|\left(\frac{df^N(x)}{dx}\right)_{x=x_0}\right| \quad \Rightarrow \quad \lambda = \lim_{N\to\infty}\frac{1}{N}\log\left|\prod_{i=0}^{N-1} f'(x_i)\right| \quad \Rightarrow$$

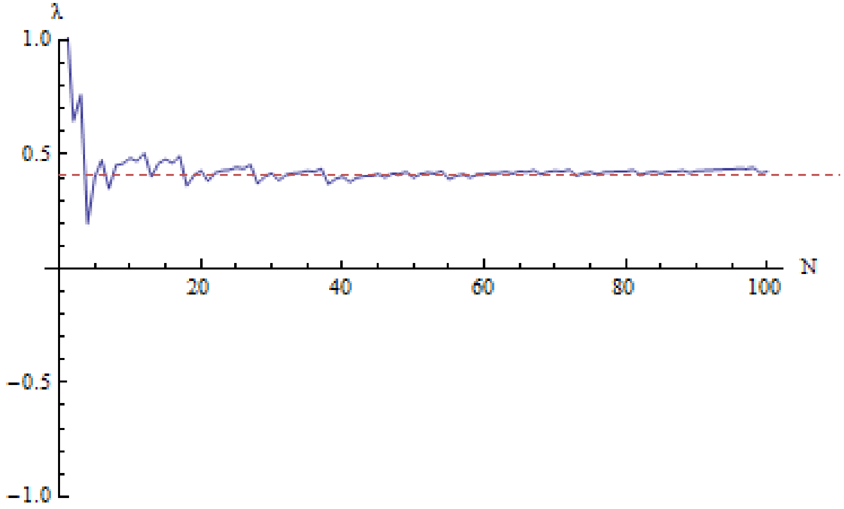
$$\boxed{\lambda(x_0) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\log|f'(x_i)|}$$

$\lambda<0$: shrinking $\rightarrow$ attraction
$\lambda>0$: expanding $\rightarrow$ chaos

Lyapunov exponent for logistic map

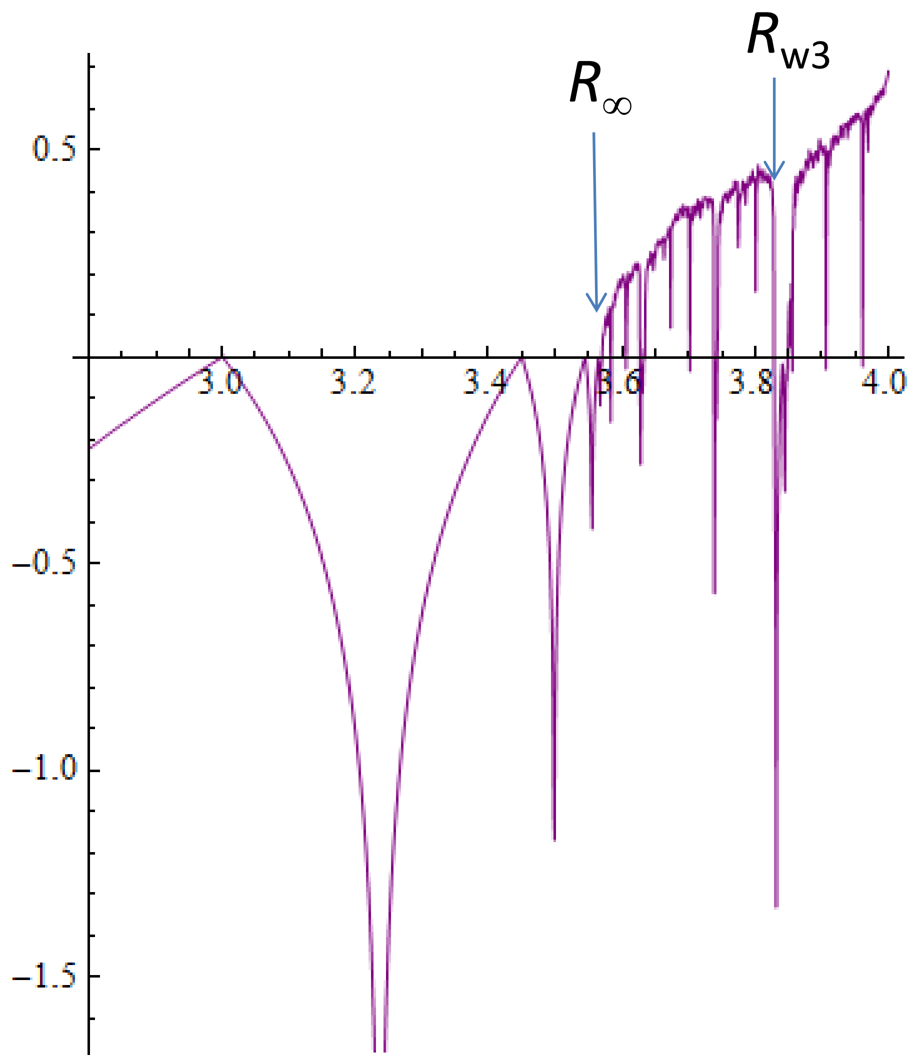


r=3.1



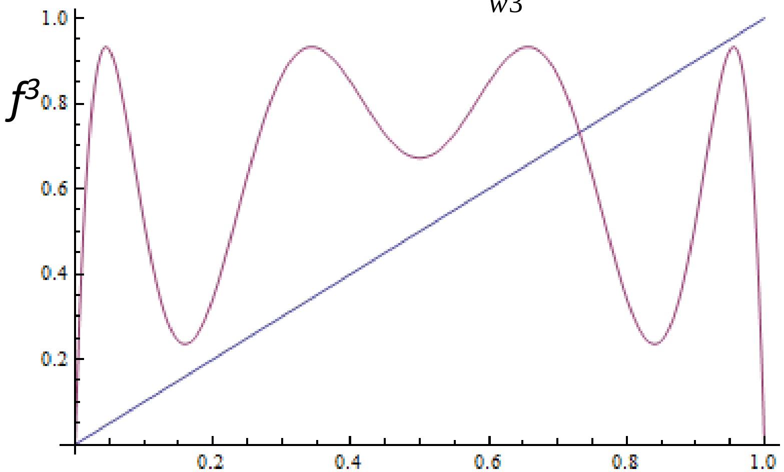
r=3.8

Lyapunov exponent for logistic map

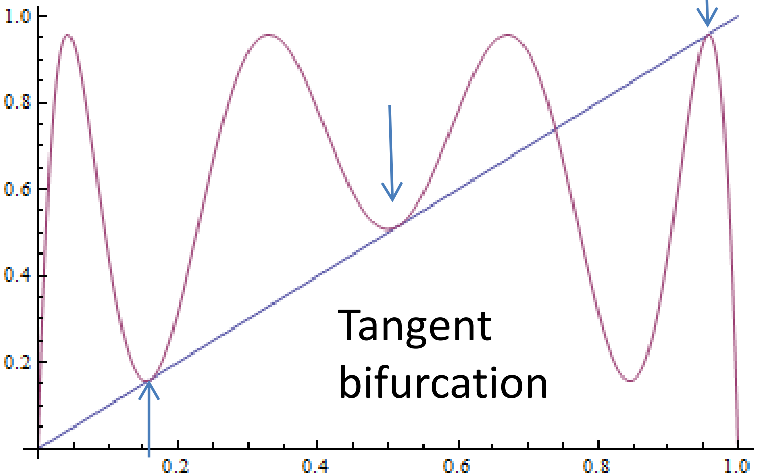


The 3-period window

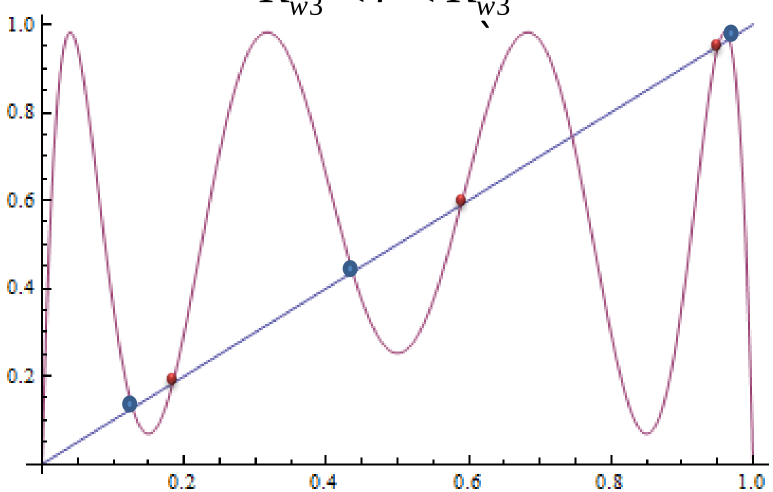
$r < R_{w3}$

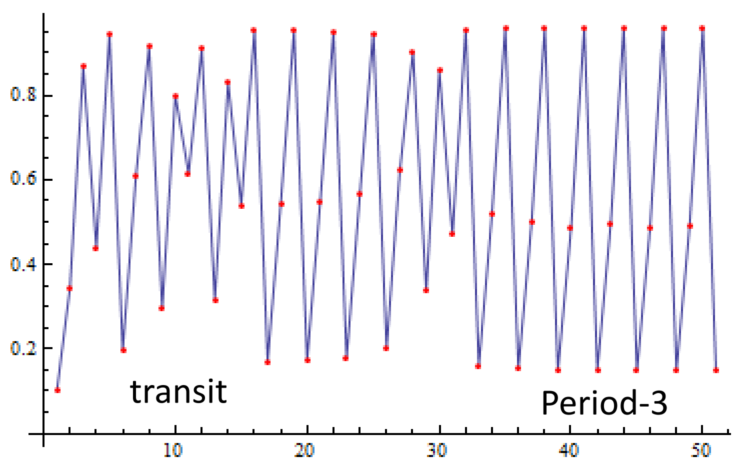


$r = R_{w3} = 1 + \sqrt{8} \approx 3.8284$



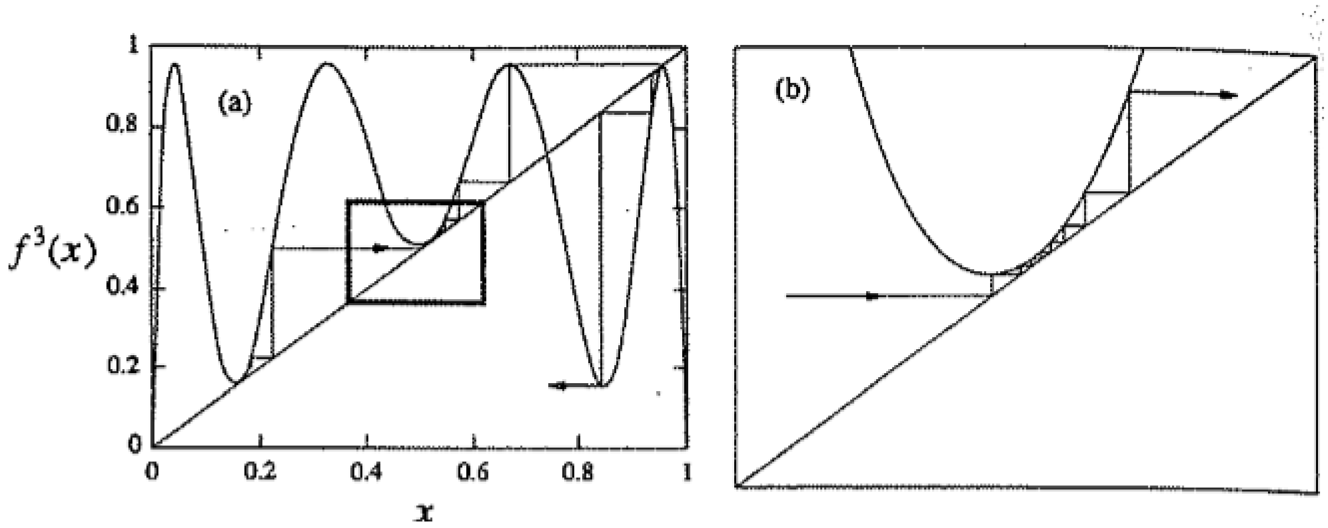
$R_{w3} < r < \bar{R}_{w3}$

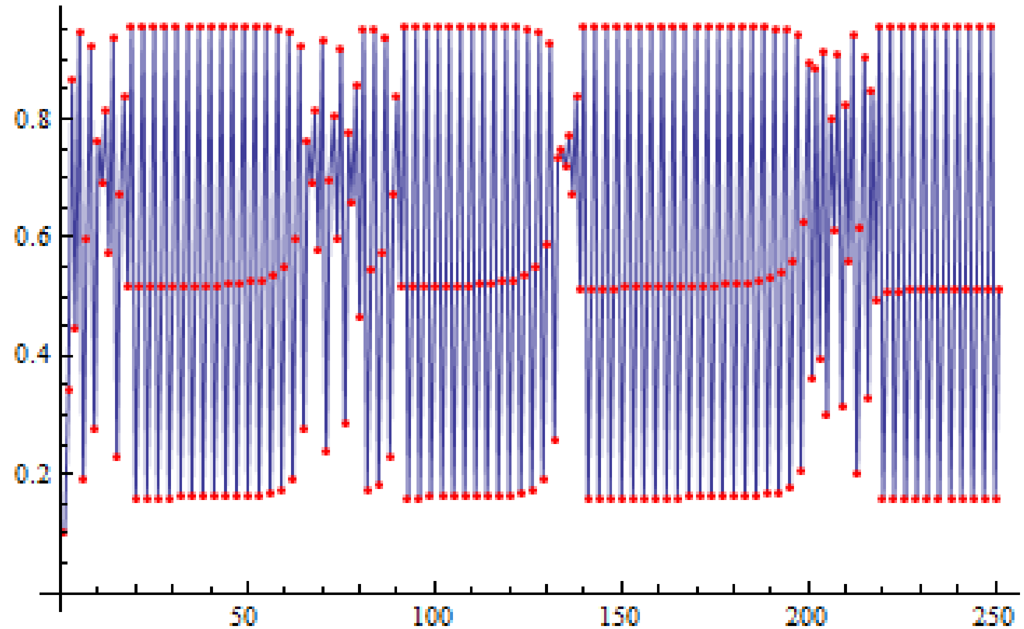


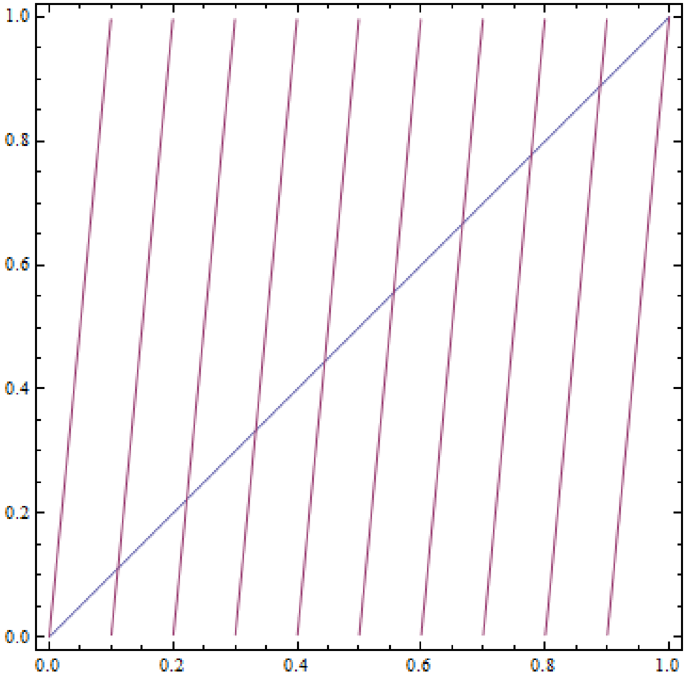


$r = R_{w3} + 0.01$

Tangent bifurcations & intermittency







The decimal shift map

$$x_{n+1} = \sigma_{10}(x_n) = 10x_n \bmod (1), \qquad x_n \in [0,1)$$

a) the operation (mod 1)

$x = 1,234567890123456.. \rightarrow x = 0,234567890123456..$

b) the operation × 10

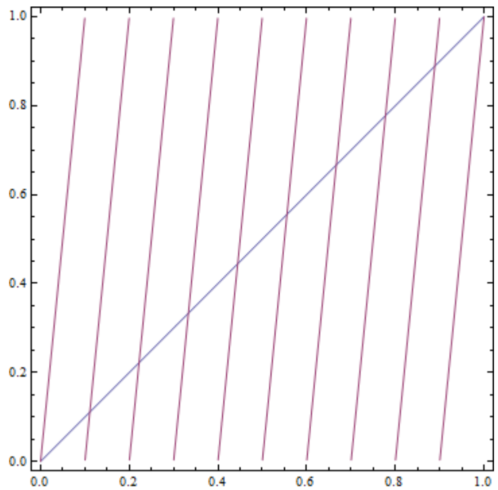
$x = 1,234567890123456.. \rightarrow x = 12,34567890123456..$

c) Complete operation

$$x_0 = 0,123456789012345...$$

$$x_1 = 0.23456789012345?..$$

$$x_{14} = 0.5???????????????????$$



The decimal shift map

$$x_{n+1} = \sigma_{10}(x_n) = 10x_n \bmod (1), \qquad x_n \in [0,1)$$

Exponential divergence of initial conditions

$x_{01} =$ 0,12345678901234**56**.. $\quad x_{02} =$ 0,12345678901234**67**.. $\quad$ (d $= 10^{-15}$)

$x_{11} =$ 0,2345678901234**56**... $\quad x_{12} =$ 0,2345678901234**67**... $\quad$ (d $= 10^{-14}$)

$x_{02} =$ 0,345678901234**56**.... $\quad x_{02} =$ 0,345678901234**67**.... $\quad$ (d $= 10^{-13}$)

....

$x_{91} =$ 0,01234**56**........... $\quad x_{92} =$ 0,01234**67**............ $\quad$ (d $= 10^{-6}$)

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log\left| f\,'(x_i) \right| = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log(10) = \log(10)$$

some definitions

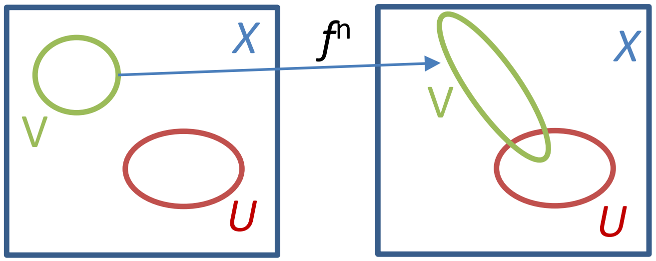
- Let Y is a subset of X and a metric $d(x,y)$. The following statements are equivalent :
  1) Y is **dense** in X
  2) For any $x \in X$ and $\varepsilon>0$, there exist $y \in Y$ such that $d(x,y)<\varepsilon$
  3) For any $x \in X$, there exist a sequence $\{y_n\}$ in Y that converges to x.

- An invariant map f: $X \rightarrow X$ is **transitive** if

$$\forall \quad U \in X, \quad V \in X, \quad U \cap V = \varnothing$$

$$\Rightarrow \quad \exists n \quad s.t. \quad U \cap f^n(V) \neq \varnothing$$

*If $\quad U \cap f^n(V) \neq \varnothing \quad \forall n > n_0 \quad \Rightarrow \quad mixing$



theorem : The map f: $X \rightarrow X$ is **transitive** if it has **a dense orbit**

- An invariant map f: $X \rightarrow X$ possesses **sensitive dependence of initial conditions** if for any $x_0 \in X$ and $\varepsilon>0$ there exist $\delta>\varepsilon$ and $y_0 \in X$ such that :

$$\text{if} \quad |x_0 - y_0| < \varepsilon, \quad \exists k > 0 \quad \text{such that} \quad d(f^k(x_0), f^k(y_0)) > \delta$$

examples

- $x_{n+1} = 2x_n \qquad (x_n = 2^n x_0), \qquad x \in R$

  Let $|x_0\text{-}y_0|<\varepsilon$ and $\delta=2^k\varepsilon$, $k>0$ (so $\delta>\varepsilon$)

  $$| f^n(x_0) - f^n(y_0) | = 2^n | x_0 - y_0 | = 2^n \varepsilon > \delta \quad \forall n > k \qquad \Rightarrow$$

  Sensitivity to initial conditions

  Let $U=(3,4)$, $V=(0,1)$, if $x_0 \in U$ then

  $$x_n \in U' = (3\cdot 2^n, 4\cdot 2^n) \quad \text{and} \quad U' \cap V = \varnothing \qquad \Rightarrow$$

  no transitivity

examples

- $x_{n+1} = x_n + c \quad \mathrm{mod}(1), \quad x, c \in R, \quad 0 < c < 1, \quad x_n = x_0 + nc \ \ \mathrm{mod}(1)$

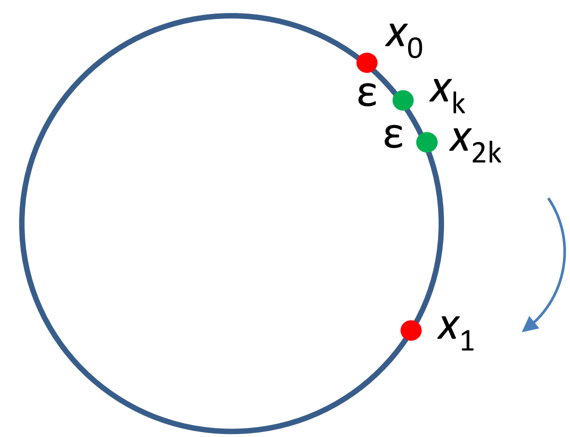
ι) $c = \dfrac{m}{k} \in Q \quad \Rightarrow \quad x_k = x_0 + m \quad \mathrm{mod}(1) \quad \Rightarrow \quad x_k = x_0$ periodic trajectory

$\text{if} \quad |x_0 - y_0| = \varepsilon \quad \Rightarrow \quad |x_k - y_k| = |x_0 - y_0| = \varepsilon$ no sensitivity to initial conditions

ιι) $c \in R \setminus Q. \quad \text{Let} \quad |c - \dfrac{m}{k}| < \varepsilon' \quad \Rightarrow \quad |kc - m| < k\varepsilon' = \varepsilon$

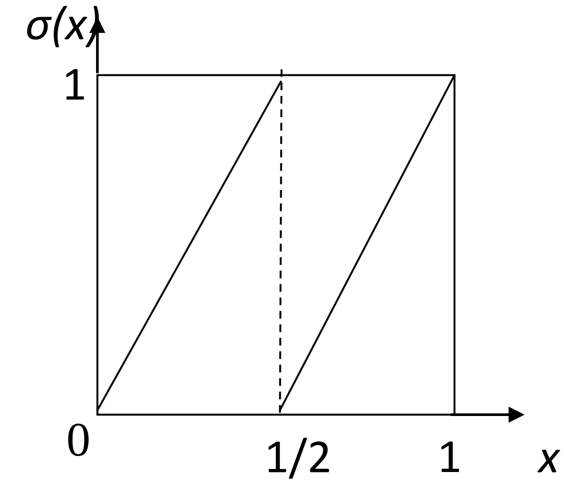
$$x_k = x_0 + kc \ \mathrm{mod}(1) = x_0 + kc - m \ \mathrm{mod}(1) \quad \Rightarrow \quad |x_k - x_0| < \varepsilon$$

$$|x_k - x_0| = |x_{2k} - x_k| = \ldots. < \varepsilon$$



dense trajectory
no sensitivity to initial conditions

The Bernoulli Shift and Symbolic Dynamics



$$x_{n+1} = \sigma(x_n) = 2x_n \bmod 1, \qquad x_n \in [0,1)$$

$$x = a_0, a_1 a_2 a_3 \ldots\ldots a_N a_{N+1} \ldots\ldots \quad , \quad a_i \in \{0,1\}$$

or

$$\Sigma = \left\{ \{a_n\}_{n=0}^{\infty} : \quad a_n = 0 \quad \text{or} \quad 1 \right\}$$

e.g. $\{0\}, \quad \{1\}, \quad \{01\}, \quad \{10\}, \quad \{100101\}, \quad \{\overline{01011}\}, \quad \{1001101\overline{10}\} \quad \in \Sigma$

- **Distance** between $x=\{x_0,x_1,x_2\ldots\} \in \Sigma$ and $y=\{y_0,y_1,y_2\ldots\} \in \Sigma$

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$$

- **The shift map**

$$\sigma : \Sigma \to \Sigma, \qquad \sigma(\{a_0 a_1 a_2 \ldots\}) = \{a_1 a_2 a_3 \ldots\}$$

Three properties of the shift map

$$\sigma : \Sigma \to \Sigma, \qquad \sigma(\{a_0 a_1 a_2 ...\}) = \{a_1 a_2 a_3 ...\} \qquad \Sigma = \left\{ \{a_n\}_{n=0}^{\infty} : \quad a_n = 0 \quad \text{or} \quad 1 \right\}$$

1. Periodic orbits are dense in Σ

Let $s = \{a_0, a_1, a_2, a_3, a_4, \ldots, a_n, a_{n+1}, \ldots\}$ any sequence of Σ.

We construct the following periodic orbits

$$p_1 = \{a_0, a_0, a_0, ...\} = \{\overline{a_0}\} \quad \to \quad d(p_1, s) \le 2^{-1}$$

$$p_2 = \{a_0 a_1 a_0 a_1, ...\} = \{\overline{a_0 a_1}\} \quad \to \quad d(p_1, s) \le 2^{-2}$$

....

$$p_n = \{a_0 a_1 a_2 ... a_n, a_0 a_1 a_2 ... a_n, ...\} = \{\overline{a_0 a_1 ... a_n}\} \quad \to \quad d(p_n, s) \le 2^{-n}$$

....

Three properties of the shift map

$$\sigma : \Sigma \to \Sigma, \qquad \sigma(\{a_0 a_1 a_2 \ldots\}) = \{a_1 a_2 a_3 \ldots\} \qquad \Sigma = \left\{ \{a_n\}_{n=0}^{\infty} : \quad a_n = 0 \quad \text{or} \quad 1 \right\}$$

2. The shift map is **transitive** or, equivalently, it has one dense orbit in Σ

We construct the following sequence that belongs to Σ

$$s^* = \{ \underbrace{0\,1}_{\text{singles}} \ \underbrace{00\,01\,10\,11}_{\text{doubles}}\ \underbrace{000\,001\,010\,100\,011 \ldots}_{\text{triples}} \quad \underbrace{\ldots\ldots}_{n-ples} \quad \}$$

i.e. s^* contains segments of all possible elements of Σ with length n and, consequently,

$$\forall s \in \Sigma, \quad \forall k \quad \exists n \quad \text{such that} \quad d(f^n(s^*), s) \le 2^{-k}$$

Three properties of the shift map

$$\sigma : \Sigma \to \Sigma, \qquad \sigma(\{a_0 a_1 a_2 \ldots\}) = \{a_1 a_2 a_3 \ldots\} \qquad \Sigma = \left\{ \{a_n\}_{n=0}^{\infty} : \quad a_n = 0 \quad \text{or} \quad 1 \right\}$$

3. The shift map is sensitive to initial conditions

$$x_0 = a_0 a_1 a_2 \ldots a_k a_{k+1} a_{k+2} \ldots\ldots \qquad y_0 = a_0 a_1 a_2 \ldots a_k b_1 b_2 \ldots\ldots \quad (b_1 \neq a_{k+1}) \qquad d(x_0, y_0) = 2^{-k}$$

$$x_1 = \sigma(x_0) = a_1 a_2 \ldots a_k a_{k+1} a_{k+2} \ldots\ldots \quad y_1 = \sigma(y_0) = a_1 a_2 \ldots a_k b_1 b_2 \ldots\ldots \qquad d(\sigma(x_0), \sigma(y_0)) = 2^{-k+1}$$

$$x_2 = \sigma^2(x_0) = a_2 \ldots a_k a_{k+1} a_{k+2} \ldots\ldots \quad y_2 = \sigma^2(y_0) = a_1 a_2 \ldots a_k b_1 b_2 \ldots\ldots \quad d(\sigma^2(x_0), \sigma^2(y_0)) = 2^{-k+2}$$

....

- Lyapunov exponent $\quad \lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log\left|\sigma'(x_i)\right| = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log(2) = \log(2)$

Devaney's definition of chaos

An invariant map f: $X \to X$ is **chaotic** if it has the following properties

1. The map is transitive in an invariant set S if

2. The periodic orbits of the map form a dense set in S

3. The map possesses sensitivity in initial conditions.

- The shift map σ is chaotic
- Any map f: $X \to X$ which is **topologically conjugate** to σ is chaotic

>> A map f: $X \to X$ is topologically conjugate to the map g: $Y \to Y$ if there exist a homeomorphism $h: X \to Y$ such that

$$f(h(x)) = h(g(x)) \quad \text{or} \quad g(x) = h^{-1}(f(h(x)))$$

The logistic map for *r*=4

$$x_{n+1} = 4x_n(1-x_n)$$

$$x = \frac{1-\cos(\pi\theta)}{2}, \quad 0 \le \theta \le 1 \qquad \Rightarrow$$

$$\frac{1-\cos(\pi\theta_{n+1})}{2} = 4\frac{1-\cos(\pi\theta_{n+1})}{2}\left(1-\frac{1-\cos(\pi\theta_{n+1})}{2}\right) \quad \Rightarrow \quad 1-\cos(\pi\theta_{n+1}) = 2-2\cos(\pi\theta_n)$$

$$\Rightarrow \quad \cos(\pi\theta_{n+1}) = \cos(2\pi\theta_n) \qquad \Rightarrow \qquad \theta_{n+1} = 2\theta_n \ \mathrm{mod}(1)$$

- The logistic map for *r*=4 is conjugate to the shift map, hence is chaotic

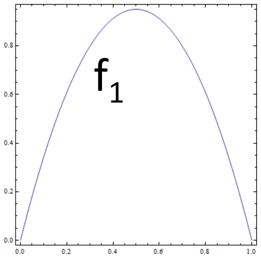
Universal behavior of quadratic maps

A continuously differentiable map *f* which

1) maps the interval [*a,b*] to itself
2) It has a single maximum in [*a,b*]
3) The Schwartzian derivative is negative in whole interval [*a,b*]

$$S_f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

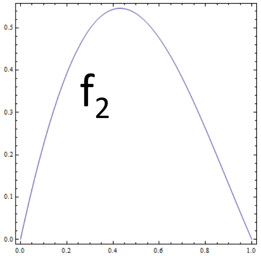
displays an infinite sequence of period doubling bifurcations with the same constants ***a*** and ***δ***.



$\longrightarrow f_1^n(x)$

r=r$_\infty$ n$\rightarrow\infty$ g(x)

universal function



$\longrightarrow f_2^n(x)$

composition law

$$g(y) = -ag\left(g(-y/a)\right)$$

$$(y = x - x_c)$$

maps for exercise

$$x_{n+1} = r + x_n - x_n^3$$

$$x_{n+1} = r x_n e^{-x_n}$$

$$x_{n+1} = r\cos(\pi x_n)$$